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The Determination of the Scattering Potential from the Spectral Measure Function

IRVIN KAY and HARRY E. MOSES

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THE DETERMINATION OF THE SCATTERING POTENTIAL FROM THE SPECTRAL MEASURE FUNCTION

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Abstract

By generalizing the method of Gelfand and Levitan it is shown that in many cases the potential can be obtained uniquely from the spectral measure function, if we specify the asymptotic behavior in some representation of the eigenstates of the total Hamiltonian H associated with the measure function. In contrast to earlier treatments where one restricts oneself to an unperturbed Hamiltonian H_o and a representation such that the H_o is a second derivative operator and the potential is diagonal, the procedure described is quite general: H_o may have almost any character, and the potential need not be diagonal in the representation in which the asymptotic description of the eigenfunctions is specified.

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1. Introduction

For the case of the radial equation Gelfand and Levitan[1] have shown how the scattering potential can be obtained from the spectral measure function. By showing, in addition, how the spectral measure function can be obtained from the scattering phases Jost and Kohn[2] and Levinson[3] have been able to construct the scattering potential from the phases. Kay[4] and Corinaldesi[5] have adapted the method of references [1] and [2] to the one-dimensional wave equation ($-\infty < x < +\infty$) and the radial Gordon-Klein-Schroedinger equation respectively. In the treatments discussed above the unperturbed Hamiltonian H_0 has been given explicitly as the kinetic energy operator, and it has been assumed that the perturbation is diagonal in the x-representation.

In the present paper we shall show how, in many cases, the potential can be obtained from the spectral measure function if we specify the asymptotic behavior of the eigenfunctions of the total Hamiltonian. The procedure which we shall use will be very general and more abstract than those used previously. We shall assume some very general properties of the unperturbed Hamiltonians instead of restricting ourselves to kinetic energy operators. Furthermore we shall not require that the perturbation be diagonal in a specific representation. It is hoped that in the future the results which we present here can be used to infer potentials from scattering operators in an abstract way.

It is found that in the abstract formulation of the problem the derivative of the spectral measure function, which we call the spectral weight function, plays a more direct role than the measure function itself. In the first part of our paper we shall discuss the problem of obtaining the weight function and its properties from the potential, and the asymptotic form of the eigenfunctions of H, while in the latter part we shall consider the inverse problem, namely to obtain the potential from the weight function and the conditions under which the potential will be unique. We shall limit ourselves to the case where H and H_o, the perturbed and unperturbed Hamiltonians, have the same spectrum, which is taken to be continuous throughout its entire range. However, we shall append some remarks as to the nature of the generalization required if we are to assume that H has point eigenvalues in addition to a continuous spectrum which coincides with the spectrum of H_o. We hope to carry out the generalization in detail in a subsequent paper.

2. The Eigenfunctions of the Unperturbed Hamiltonian

We have assumed that the unperturbed Hamiltonian H_o has a continuous spectrum only. In most cases the spectrum will be degenerate. It will therefore be convenient to introduce a set of commuting variables, collectively denoted by A_o, each of which also commutes with H_o, such that A_o and H_o form a complete set of commuting variables. Using a modification of Dirac's notation we denote the state which is simultaneously an eigenstate of H_o and A_o corresponding to the eigenvalues E and a of H_o and A_o respectively by $|H_o,A_o;E,a>$. An example of A_o occurs in the case where H_o is the kinetic energy operator. A_o could then be chosen to be the operator whose eigenvalues give the direction of the momentum.

We have

$$(2.1) \qquad H_o\,|H_o,A_o;E,a> = E|H_o,A_o;E,a>.$$

If we assume that the eigenfunctions are normalized, we have

$$(2.2) \qquad <H_o,A_o;E,a|H_o,A_o;F,b> = \delta(E-F)\delta(a,b),$$

where $\delta(a,b)$ is a generalized δ-function defined by

$$\int f(a)\delta(a,b)da = f(b) \quad \text{if the range of integration includes b} \tag{2.3}$$
$$= 0 \quad \text{otherwise .}$$

(The integration is to be understood as summation over the values of a in the discrete spectrum.) Since the set of eigenfunctions is to be a complete set, we have

$$\iint |H_o,A_o;E,a > dE\ da < H_o,A_o;E,a| = I, \tag{2.4}$$

where I is identity operator.

3. The Eigenfunctions of the Perturbed Hamiltonian and the Transformation Operator

The perturbed Hamiltonian H is given by

$$H = H_o + \varepsilon V \tag{3.1}$$

where V is the scattering potential and ε is a smallness parameter. We have assumed that the spectrum of H is the same as H_o. Hence the degeneracy of the spectrum of H is the same as that of H_o. We therefore introduce a set of commuting operators A, analogous to A_o, which together with H form a complete set of commuting variables. We denote the eigenfunction of H and A which belongs to the eigenvalues E and a of H and A respectively by $|H,A;E,a>$. Hence

$$H|H,A;E,a> = E|H,A;E,a> . \tag{3.2}$$

We also introduce the transformation operator U defined by

$$|H,A;E,a> = U|H_o,A_o;E,a> . \tag{3.3}$$

(This operator is also used, for example, by Friedrichs[6].) We now proceed to develop some relations which U must satisfy. Substituting (3.3) into (3.2) we have

(3.4)
$$\begin{aligned} H|H,A;E,a> &= EU|H_0,A_0;E,a> \\ &= UE|H_0,A_0;E,a> \\ &= UH_0|H_0,A_0;E,a> \end{aligned}$$

or

(3.5)
$$HU|H_0,A_0;E,a> = UH_0|H_0,A_0;E,a>.$$

Hence on using (2.4) we have

(3.6)
$$HU = UH_0.$$

For any function f(H) of H we infer

(3.7)
$$f(H)U = Uf(H_0).$$

It will be shown later that U has an inverse, U^{-1}. From (3.6), then,

(3.8)
$$H = UH_0U^{-1}$$

and we see that the transformation U is actually a canonical transformation between H and H_0. It is further seen that a relation like (3.8) can hold only when the spectrum of H and H_0 are the same.

The operator U or equivalently the eigenfunctions $|H,A;E,a> = U|H_0,A_0;E,a>$ are not unique. All possible solutions of (3.6) satisfy the integral equation

(3.9)
$$U = L + \varepsilon\int \frac{P}{E - H_0}\, VU\delta(E-H_0)dE,$$

where L is an arbitrary operator which commutes with the H_0. The symbol P means 'principal part' in the integration, the range of integration is the range of the spectrum of E. The operator $\delta(E-H_0)$ is defined by

(3.10)
$$\delta(E-H_0) = \int |H_0,A_0;E,a>da<H_0,A_0;E,a|,$$

and has the properties

(3.11)
$$H_0\delta(E-H_0) = \delta(E-H_0)H_0 = E\delta(E-H_0),$$

(3.12)
$$\int\delta(E-H_0)dE = I.$$

Equation (3.12) follows from (2.4). To show that U as given by (3.9) satisfies (3.6) for any operator L which commutes with H_o, we rewrite (3.6) as

(3.13) $$UH_o - H_oU = \varepsilon VU$$

which follows from (3.1). But on using (3.9) and (3.11) we find that

(3.14) $$\begin{aligned} UH_o &= LH_o + \varepsilon \int \frac{P}{E - H_o} VU\delta(E-H_o)H_o dE \\ &= H_oL + \varepsilon \int \frac{P}{E - H_o} EVU\delta(E-H_o)dE. \end{aligned}$$

and

(3.15) $$H_oU = H_oL + \varepsilon \int \frac{P}{E - H_o} H_oVU\delta(E-H_o)dE.$$

Hence

(3.16) $$\begin{aligned} UH_o-H_oU &= \varepsilon \int \frac{P}{E - H_o} (E-H_o)VU\delta(E-H_o)dE \\ &= \varepsilon \int VU\delta(E-H_o)dE = \varepsilon VU \int \delta(E-H_o)dE = \varepsilon VU \end{aligned}$$

which is just equation (3.13).

Having introduced operators which commute with H_o, we shall show that these operators are, in a sense, diagonal in the H_o representation. Let B be any operator which commutes with H_o; then

(3.17) $$BH_o - H_oB = 0$$

or

(3.18) $$\begin{aligned} &< H_o,A_o;E,a|BH_o|H_o,A_o;F,b > \\ &- < H_o,A_o;E,a|H_oB|H_o,A_o;F,b > \\ &= (F-F) < H_o,A_o;E,a|B|H_o,A_o;F,b > \\ &= 0. \end{aligned}$$

From the second equation of (3.18) it follows that

$$< H_o,A_o;E,a|B|H_o,A_o;F,b > = < a|\beta(E)|b > \delta(E-F) \tag{3.19}$$

where $< a|\beta(E)|b >$ is a function of a, b and E (a proper function, i.e., not a δ-function, in terms of E) which can be obtained from the operator B. The δ-function on the right of equation (3.19) shows that B is diagonal in the H_o representation.

4. The Outgoing and Incoming Eigenfunctions: The Scattering Operator

There are two transformation operators which play a particularly important role. We denote them by $U_{\pm}$. The integral equations for these operators are obtained by choosing the operator L properly. The operator U_- is required to satisfy the condition

$$\lim_{t \to -\infty} e^{iH_o t} e^{-iHt} U_- |\emptyset > = |\emptyset > \tag{4.1}$$

where $|\emptyset >$ is a state in Hilbert space, that is, quadratically integrable. The meaning of (4.1) is that if we choose the solution $|\emptyset(t) >$ of the perturbed Schroedinger equation

$$H|\emptyset(t) > = i \frac{\partial}{\partial t} |\emptyset(t) > \tag{4.2}$$

to behave like

$$|\emptyset(t) > \cong e^{-iH_o t} |\emptyset > \tag{4.3}$$

when t -> -∞, then at finite times

$$|\emptyset(t) > = e^{-iHt} U_- |\emptyset > . \tag{4.4}$$

Since the Schroedinger equation has only a first derivative in time, only one initial condition is required to obtain a unique solution. Equation (4.4) may be regarded as the solution of (4.2) with the initial condition (4.3).

In Appendix I it is shown that condition (4.1) fixes the operator L as

$$L = I - i\pi\varepsilon \int \delta(E-H_0)VU_-\delta(E-H_0)dE. \tag{4.5}$$

It is to be noted that L as given by (4.5) commutes with H_0, as required. Substituting (4.5) into (3.9) we find

$$U_- = I + \varepsilon \int \gamma_-(E-H_0)VU_-\delta(E-H_0)dE, \tag{4.6}$$

where

$$\gamma_-(x) = -i\pi\delta(x) + \frac{P}{x} = \lim_{\zeta \to +0} \frac{1}{x+i\zeta} \ . \tag{4.7}$$

The eigenfunctions $|H,A;E,a >_-$ of H given by

$$|H,A;E,a >_- = U_-|H_0,A_0;E,a > \tag{4.8}$$

are generally called the 'outgoing waves'. (Usually, however, the plus subscript is used to denote these eigenfunctions.)

The scattering operator can be conveniently expressed in terms of U_-. It is defined in the following way: If $|\phi(t) >$ is a solution of the Schroedinger equation (4.2) such that

$$\lim_{t \to \infty} e^{iH_0t}|\phi(t) > = |\phi > \tag{4.9}$$

exists, then the scattering operator S is given by

$$\lim_{t \to +\infty} e^{iH_0t}|\phi(t) > = S|\phi >. \tag{4.10}$$

If we identify $|\phi(t) >$ with $e^{-iHt}U_-|\phi >$ then the scattering operator is given by

$$\lim_{t \to +\infty} e^{iH_0t}e^{iHt}U_-|\phi > = S|\phi > . \tag{4.11}$$

In Appendix I, it is shown that

$$S = I - 2\pi i\varepsilon \int \delta(E-H_0)VU_-\delta(E-H_0)dE. \tag{4.12}$$

The scattering operator takes a state which behaves like the solution $e^{-iH_0t}|\phi >$

of the unperturbed equation at $t = -\infty$ into another solution $e^{-iH_0t}S|\phi\rangle$ of the unperturbed equation when $t = +\infty$.

The operator U_+ is used to solve the final value problem. Specifically, U_+ is defined as that transformation which satisfies

$$\lim_{t\to+\infty} e^{iH_0t}e^{-iHt}U_+|\phi\rangle = |\phi\rangle. \tag{4.13}$$

The state $|\phi(t)\rangle$ at finite times is then given by

$$|\phi(t)\rangle = e^{-iHt}\,U_+|\phi\rangle, \tag{4.14}$$

where $|\phi(t)\rangle$ behaves like $e^{-iH_0t}|\phi\rangle$ for $t = +\infty$. In Appendix I it is shown that in this case the operator L must be taken as

$$L = I + i\pi\varepsilon\int\delta(E-H_0)VU_+\delta(E-H_0)dE. \tag{4.15}$$

Hence U_+ satisfies the equation

$$U_+ = I + \varepsilon\int\gamma_+(E-H_0)VU_+\delta(E-H_0)dE, \tag{4.16}$$

where

$$\gamma_+(x) = i\pi\delta(x) + \frac{P}{x} = \lim_{\zeta\to+0}\frac{1}{x-i\zeta}. \tag{4.17}$$

The eigenstates $|H,A;E,a\rangle_+$ of H given by

$$|H,A;E,a\rangle_+ = U_+|H_0,A_0;E,a\rangle \tag{4.18}$$

are usually called the 'incoming waves'. From the definition of the scattering operator it is clear that the inverse scattering operator S^{-1} is given by

$$\lim_{t\to-\infty} e^{iH_0t}e^{-iHt}\,U_-|\phi\rangle = S^{-1}|\phi\rangle. \tag{4.19}$$

In Appendix I it is shown that

$$S^{-1} = I + 2\pi i\varepsilon\int\delta(E-H_0)VU_+\delta(E-H_0)dE. \tag{4.20}$$

Generally, the operators U are not unitary. However, the operators $U_\pm$

are. In Appendix II it is shown that

$$(4.21) \qquad U_{\pm}^{*} U_{\pm} = I$$

where the asterik means Hermitian adjoint. Equation (4.21) yields orthonormality relations for $|H,A;E,a>_{\pm}$, since we have

$$(4.22) \qquad {}_{\pm}< H,A;E,a|H,A;F,b >_{\pm} = < H_0,A_0;E,a|U_{\pm}^{*}U_{\pm}|H_0,A_0;F,b >$$

$$= < H_0,A_0;E,a|H_0,A_0;F,b >$$

$$= \delta(E-F)\delta(a,b).$$

Since we have assumed the spectrum of H is the same as that of H_0, each set of eigenfunctions, $|H,A;E,a>_{+}$ and $|H,A;E,a>_{-}$, is complete. Accordingly, any arbitrary state $|\emptyset>$ can be expanded in terms of either set. Using (4.22) the coefficients of the expansion can be found and it is seen that

$$(4.23) \qquad |\emptyset> = \iint |H,A;E,a>_{\pm} dEda \; {}_{\pm}< H,A;E,a|\emptyset>$$

or

$$(4.24) \qquad \iint |H,A;E,a>_{\pm} dEda \; {}_{\pm}< H,A;E,a| = I;$$

thus on using (4.8), (4.18), (2.4) we have

$$(4.25) \qquad U_{\pm} U_{\pm}^{*} = I .$$

Finally conditions (4.21) and (4.25) imply, by definition of inverse, that

$$(4.26) \qquad U_{\pm}^{*} = U_{\pm}^{-1}$$

which is the condition that U be unitary.

5. The General Transformation Operator and the Spectral Weight Function

We shall now consider properties of the general operator U which are analogous to the unitary property of $U_{\pm}$.

From (3.9) we see that we can write U as

$$U = M_{\pm} + \varepsilon \int \gamma_{\pm}(E-H_0)VU\delta(E-H_0)dE \tag{5.1}$$

where

$$M_{\pm} = L \mp i\pi\varepsilon \int \delta(E-H_0)VU\delta(E-H_0)dE. \tag{5.2}$$

In Appendix III we prove that

$$U = U_{\pm} M_{\pm} . \tag{5.3}$$

From Equation (5.2) it is clear that the operators $M_{\pm}$ commute with H_0. From (5.3) we can show that the operators $M_{\pm}$ have inverses. For from (5.3) we have

$$\begin{aligned} |H,A;E,a > &= U|H_0,A_0;E,a > \\ &= \iint U_{\pm}|H_0,A_0;E',a' > dE'da' \\ &\quad \cdot < H_0,A_0;E',a'|M_{\pm}|H_0,A_0;F,a >. \end{aligned} \tag{5.4}$$

But if we write (see Eq. (3.19))

$$< H_0,A_0;E',a'|M_{\pm}|H_0,A_0;E,a > = < a'|\mu_{\pm}(E)|a > \delta(E-E') \tag{5.5}$$

the above equation becomes

$$|H,A;E,a > = \int |H,A;E,a' >_{\pm} da' < a'|\mu_{\pm}(E)|a > . \tag{5.6}$$

Equations (5.6) shows that the operators $M_{\pm}$ merely take linear combinations of the eigenfunctions belonging to one eigenvalue of H into another eigenfunction of H with the same eigenvalue. It ought to be possible to expand $|H,A;E,a >_{\pm}$ in terms of $|H,A;E,a >$, since all these eigenfunctions span the same space. The operators which perform this expansion will be denoted by $M_{\pm}^{-1}$. These operators also commute with H_0, as can be shown from (5.6).

Incidentally, equation (5.3) also appears in reference [6].

As a special case of (5.1), let us take the positive sign for the subscripts and identify U with U_-. Then using (4.5) for the operator L associated with U_-, we have

(5.7) $$M_+ = I - 2i\pi\varepsilon \int \delta(E-H_0)VU_-\delta(E-H_0)dE = S.$$

Hence (5.3) gives

(5.8) $$U_- = U_+S$$

or on using (4.26)

(5.9) $$S = U_+^*U_-.$$

From (5.3) we have generally

(5.10) $$U_+M_+ = U_-M_-$$

or

$$M_+ = U_+^*U_-M_- = SM_-.$$

Hence

(5.11) $$S = M_+M_-^{-1}.$$

Equation (5.11) shows how one can construct the scattering operator from a general transformation. Equation (5.11) is a generalization of a result that often appears in the literature, in the discussions of the radial equation.

Let us define the weight operator W by

(5.12) $$W = M_+^{-1}M_+^{*-1} = M_-^{-1}\ M_-^{*-1}.$$

It is to be noted that W is a positive definite Hermitian operator which commutes with H_o.

From (5.3), (4.21), (4.25) we have

(5.13) $$WU^*U = I,$$

(5.14) $$UWU^* = I.$$

Equation (5.13) gives the orthogonality relations for $|H,A;E,a>$, and (5.14) gives the completeness relation for these eigenstates. From (5.13) and (5.14) it is clear that

(5.15) $$U^{-1} = WU^*,$$

(5.15a) $$W^{-1} = U^{*}U.$$

To justify our calling W a weight operator let us write

(5.16) $$< H_o,A_o;E,a|W|H_o,A_o;F,b > = < a|\omega(E)|b > \delta(E-F),$$

as is possible from (3.19). Then for any state $|\phi >$ we have, from (5.14) and (2.4),

(5.17) $$\begin{aligned} |\phi > &= UWU^{*}|\phi > \\ &= \iiiint U|H_o,A_o;E,a > dEda < H_o,A_o;E,a|W|H_o,A_o;F,b > \\ &\qquad \cdot dFdb < H_o,A_o;F,b|U^{*}|\phi > \\ &= \iiiint |H,A;E,a > dEda < H_o,A_o;E,a|W|H_o,A_o;F,b > \\ &\qquad \cdot dFdb < H,A;F,b|\phi >, \end{aligned}$$

or on using (5.16) we have the expansion of $|\phi >$ in terms of $|H,A;E,a >$, namely

(5.18) $$|\phi > = \int dE \iint |H,A;E,a > da < a|\omega(E)|b > db < H,A;E,a|\phi > .$$

If the spectrum of H had no degeneracy, (5.18) would be written as

(5.18a) $$|\phi > = \int dE|H;E > \omega(E) < H;E|\phi >$$

and $\omega(E)$ would play the role of the usual weight function. In the case of degeneracy the weight function must be replaced by the weight operator $< a|\omega(E)|b >$.

To summarize: In the direct problem, if H is given and a set of eigenfunctions (or equivalently a transformation operator U) is specified by choosing the appropriate boundary conditions, we can find the spectral weight operator W corresponding to those eigenfunctions.

6. The Inverse Problem

In the inverse problem we are given an operator W and seek the conditions under which we can solve the equations (cf. (5.13) and (5.14))

(6.1) $$WU^*U = I$$

(6.2) $$UWU^* = I,$$

for U such that $|H,A;E,a\rangle = U|H_0,A_0;E,a\rangle$ is an eigenfunction of a Hamiltonian $H = UH_0U^{-1}$ corresponding to a suitable choice of boundary conditions on $|H,A;E,a\rangle$. Specifically we shall rewrite (6.1) and (6.2) as

(6.3) $$WU^* = U_0$$

(6.4) $$U_0 = U^{-1}$$

and obtain theorems which will tell us under what conditions we can obtain unique solutions U, U_0 of (6.3) such that (6.4) is satisfied and such that the eigenfunctions $|H,A;E,a\rangle = U|H_0,A_0;E,a\rangle$ approach $|H_0,A_0;E,a\rangle$ asymptotically in some specified representation.

Of the above conditions we shall, in the present section, discuss the necessary conditions on W, independent of the boundary conditions on the eigenfunctions $|H,A;E,a\rangle$.

We shall now show that in order for (6.3) to have a solution for U and U_0 such that (6.4) holds, W must be a positive definite, Hermitian operator which has an inverse. Further, in order that H, as given by

(6.5) $$H = UH_0U_0 = UH_0U^{-1},$$

be Hermitian, W must commute with H_0.

Once we have proved this, we shall henceforth assume that W is a positive definite Hermitian operator which commutes with H_0 and has an inverse.

First of all let us show that W must have an inverse. From (6.3) and (6.4),

(6.6) $$WU^*U = I,$$

from which by multiplication on the right by $U^{-1}\,U^{*-1}$ we get

$$W = U^{-1}U^{*-1} = (U^{*}U)^{-1}. \tag{6.7}$$

Clearly W has an inverse W^{-1} given by

$$W^{-1} = U^{*}U. \tag{6.8}$$

That W must be Hermitian follows immediately from (6.7). We can also show from (6.7) that W is positive definite. To prove this we must show that

$$< \Phi\,|W|\,\Phi > \;= 0$$

for any state $|\,\Phi >$ which is not zero, and that

$$< \Phi\,|W|\,\Phi > = 0$$

implies $|\,\Phi > = 0$. First of all assume $< \Phi\,|W|\,\Phi > = 0$. Then we write $|\theta > = U^{*-1}|\,\Phi >$ and note that

$$< \Phi\,|W|\,\Phi > = < \Phi\,|U^{-1}U^{*-1}|\,\Phi > = 0 = < \theta|\theta >$$

implies

$$|\theta > = U^{*-1}\,|\,\Phi > = 0.$$

Multiplying by U^{*} we have $|\,\Phi > = 0$.

Let $|\,\Phi >$ be any non-zero state. Then $U^{*-1}|\,\Phi > = |\theta >$ cannot be 0, for if it were, $|\,\Phi > = U^{*}|\theta >$ would equal zero. But $< \Phi\,|W|\,\Phi > = < \theta|\theta >$ and hence is positive if $|\,\Phi >$ is a non-zero vector.

We now show that W must commute with H_o in order that H as given by (6.5) be Hermitian. From (6.5) and (6.3) we have

$$\begin{aligned} H &= UH_oU_o = UH_oWU^{*}, \\ H^{*} &= U_o^{*}\,H_oU^{*} = UWH_oU^{*}\;. \end{aligned} \tag{6.9}$$

Then if H is to equal H^{*} we see from (6.9) that the following equation must hold:

$$UH_oWU^* = UWH_oU^*,$$

On multiplying on the right by U^{*-1} and on the left by U^{-1} we have

(6.9a) $$H_oW = WH_o.$$

Hence, a necessary condition that H be Hermitian is that W commutes with H_o. This condition is also sufficient as can be easily seen.

The potential V which one would obtain is given by

(6.10) $$\varepsilon V = H-H_o = UH_oU^{-1}-H_o,$$

(6.10a) $$\varepsilon VU = UH_o-H_oU.$$

To show that eigenfunctions $|H,A;E,a>$ of H can be obtained from

(6.11) $$|H,A;E,a> = U|H_o,A_o;E,a>$$

we use (6.5). Then

(6.12) $$H|H,A;E,a> = HU|H_o,A_o;E,a> = UH_o|H_o,A_o;E,a>$$
$$= EU|H_o,A_o;E,a> = E|H,A;E,a>.$$

7. Boundary Conditions on the Eigenfunctions and Triangularity Conditions on Operators

It is clear that for a given operator W which satisfies the conditions discussed in Section 6, there are many operators U_o and U which satisfy $WU^* = U_o$ and $U^{-1} = U_o$. We wish to introduce additional conditions, understandable from a physical point of view, which will enable us to get essentially unique solutions U, U_o which satisfy $U^{-1} = U_o$. Toward this end we consider imposing boundary conditions on the eigenfunctions $|H,A;E,a>$ to make them unique.

Let us introduce a Hermitian operator Q which is itself a complete set of commuting variables. Let us denote the eigenvalues of Q by q, whose range is

$$q_0 \leq q \leq q_1.$$

In special cases q_0 or q_1 can be 0, or ± ∞. The eigenvector of Q corresponding to the eigenvalue q is denoted by $|q >$.

The boundary condition which can reasonably be imposed on $|H,A;E,a >$ is

(7.1) $$\lim_{q \to q_0} \left[< q|H,A;E,a > - < q|H_0,A_0;E,a > \right] = 0,$$

i.e., that $< q|H,A;E,a >$ approach $< q|H_0,A_0;E,a >$ asymptotically in the Q-representation. We shall now given conditions on U, U_0 such that (7.1) will often hold and U, U_0 will be unique. These conditions will henceforth be called the triangularity conditions. Let us write

(7.2) $$\begin{aligned} U &= I + \varepsilon K, \\ U_0 &= I + \varepsilon K_0; \end{aligned}$$

then

(7.3) $$< q|H,A;E,a > = < q|H_0,A_0;E,a > + \varepsilon \int_{q_0}^{q_1} < q|K|q' > da' < q'|H_0,A_0;E,a >,$$

(7.4) $$< q|H_0,A_0;E,a > = < q|H,A;E,a > + \varepsilon \int_{q_0}^{q_1} < q|K_0|q' > dq' < H,A;E,a > .$$

Let us call any operator P triangular in the Q-representation if

(7.5) $$< q|P|q' > = 0 \qquad q' > q$$

and if $< q|P|q' >$ is a bounded function for $q' \leq q$. If K and K_0 are triangular and q_0 is finite in the Q-representation, then the eigenfunctions $|H,A;E,a >$ have the asymptotic form (7.1), since then equations (7.3) and (7.4) become

(7.6) $$< q|H,A;E,a > = < q|H_0,A_0;E,a > + \varepsilon \int_{q_0}^{q} < q|K|q' > dq' < q'|H_0,A_0;E,a >,$$

$$(7.7) \qquad < q|H_o,A_o;E,a > = < q|H,A;E,a > + \varepsilon \int_{q_o}^{q} < q|K_o|q' > dq' < q'|H,A;E,a >.$$

If $q_o = -\infty$, some additional restrictions on $< q|K|q' >$ and $< q|K_o|q' >$ as $q \to -\infty$ are needed in order that the triangularity condition guarantee the asymptotic form (7.1). An example of such a condition (a rather severe one) is that $< q|K|q' >$ and $< q|K_o|q' >$ vanish when $q < q_{min}$, where q_{min} is finite. In Section 17 we shall discuss sufficient conditions which W must satisfy in order that K and K_o be such that (7.1) holds.

In our present treatment we have not specified in which representation K, K_o is to be triangular. Our attitude is that physical reasoning should be used to tell us in which representation the asymptotic relation (7.1) is to hold and hence in which representation K, K_o are to be triangular. In contrast to this attitude, previous authors choose the Q-representation in such a way that the potential V is diagonal in that representation. The emphasis in their papers is on the proof that K and K_o are triangular in this representation.

Most of the remaining sections of the present paper are devoted to theorems which tell us under what conditions U, U_o can be obtained uniquely once the Q-representation is given in which K, K_o is to be triangular.

A summary of the theorems is the following:

1) A necessary condition for a solution is that $< q|W-I|q' >$ be a bounded function of q' in the neighborhood of $q' = q$.

2) A sufficient condition for a solution U, U_o of $WU^* = U_o$ is that the operator (W-I) be bounded less than 1.

3) A sufficient condition for the uniqueness of the solution U, U_o of $WU^* = U_o$ is that $< q|K|q' >$ and $< q|K_o|q' >$ be bounded functions in every interval. (This condition is automatically satisfied by the definition of triangularity of K, K_o in the Q-representation.)

4) A sufficient condition that $U^{-1} = U_0$ is that either $< q|K|q' >$ or $< q|K_0|q' >$ be a bounded continuous function of its arguments.

5) An alternative sufficient condition that U, U_0 are unique and $U^{-1} = U_0$ is that $(W-I)$ be bounded less than 1 and $(W-I)$ have an inverse.

8. Solution of the Equation for U, U_0

The triangularity conditions (7.5) as applied to K, K_0 enable us to find a relation between U and W, from which, under suitable conditions on W, we can find U explicitly in terms of W.

Let us write

$$W = I + \varepsilon \Omega \quad . \tag{8.1}$$

From (6.3) we have, using the fact that W is Hermitian,

$$UW = U_0^* \; . \tag{8.2}$$

Upon substituting (7.2) and (8.1) into (8.2), we obtain

$$K = K_0^* - \Omega - \varepsilon K \Omega \tag{8.3}$$

or, in terms of the Q-representation,

$$< q|K|q' > = < q|K_0^*|q' > - < q|\Omega|q' > - \varepsilon \int_{q_0}^{q} < q|K|q'' > dq'' < q''|\Omega|q' > \; . \tag{8.4}$$

In the integral occuring in (8.4) we have used the triangularity property of K.

Now let us take $q' < q$. Since $< q|K_0^*|q' > = 0$, we have

$$< q|K|q' > = -< q|\Omega|q' > - \varepsilon \int_{q_0}^{q} < q|K|q'' > dq'' < q''|\Omega|q' > \text{ for } q' < q. \tag{8.5}$$

Equation (8.5) is our generalization of the Gelfand-Levitan equation. It is an integral equation of the Fredholm type (if q is considered fixed) which gives K in terms of Ω or equivalently U in terms of W.

One could iterate (8.5) to obtain a solution for $< q|K|q' >$. However, we shall proceed in a different way by rewriting equation (8.5) in terms of an operator notation.

Let us define the operator $\delta(q-Q)$ by

(8.6) $$\delta(q-Q)|q' > = \delta(q-q')|q' > ,$$

from which it can be seen that on using

$$\int_{q_0}^{q_1} |q > dq < q| = I ,$$

we have

(8.7) $$\delta(q-Q) = |q > < q|.$$

We define the operator $\eta(q-Q)$ by

(8.8) $$\eta(q-Q)|q' > = \eta(q-q')|q' > ,$$

where $\eta(q)$ is the Heaviside step function

(8.9) $$\begin{aligned} \eta(q) &= 1, \qquad q > 0, \\ &= 0, \qquad q < 0. \end{aligned}$$

Any operator R, say, which is triangular in the sense

(8.10) $$< q|R|q' > = 0, \qquad q' > q$$

together with the boundedness condition can be represented as

(8.11) $$< q|R|q' > = < q|P|q' > \eta(q-q'),$$

where

(8.12) $$< q|P|q' > = < q|R|q' > \quad \text{for } q' \leq q$$

while

$$< q|P|q' > \text{ is arbitrary for } q' > q.$$

Relation (8.11) can be written

(8.13) $$R = \int_{q_0}^{q_1} \delta(q-Q)P\,\eta(q-Q)dq = \int_{q_0}^{q_1} \eta(Q-q)P\delta(q-Q)dq ,$$

where as before P is an operator such that

(8.14) $$< q|P|q' > = < q|R|q' >, \qquad \text{for } q < q'$$

and

$$< q|P|q' > \text{ is arbitrary for } q > q'.$$

In particular P can be taken as R. Let us multiply each side of equation (8.5) by $\eta(q-q')$. Since $< q|K|q' > \eta(q-q') = < q|K|q' >$ we have

(8.15) $$< q|K|q' > = -< q|\Omega|q' > \eta(q-q') - \varepsilon \int_{q_0}^{q_1} < q|K|q'' > \eta(q-q'')dq'' < q''|\Omega|q' > \eta(q-q').$$

In order that $< q|K|q' >$ have a solution it is necessary that $< q|\Omega|q' > \eta(q-q')$ exist as the kernel of an integral operator. We shall discuss this point more fully later. From (8.15) we get, in an abstract notation,

(8.16) $$K = -\int_{q_0}^{q_1} \delta(q-Q)\Omega\,\eta(q-Q)dq - \varepsilon \int_{q_0}^{q_1} \delta(q-Q)K\,\eta(q-Q)\Omega\,\eta(q-Q)dq.$$

We regard (8.16) as our equation for K. We maintain that if the operator $[I + \varepsilon\Omega\,\eta(q-Q)]^{-1}$ exists, then a solution of (8.16) is

(8.17) $$K = -\int_{q_0}^{q_1} \delta(q-Q)[I + \varepsilon\Omega\,\eta(q-Q)]^{-1}\Omega\eta(q-Q)dq.$$

This can be shown as follows. If $[I + \varepsilon\Omega\eta(q-Q)]^{-1}$ exists we have

(8.18) $$[I + \varepsilon\Omega\eta(q-Q)]^{-1}[I + \varepsilon\Omega\eta(q-Q)] = [I + \varepsilon\Omega\eta(q-Q)][I + \varepsilon\Omega\eta(q-Q)]^{-1} = I,$$

from which

(8.19) $$[I + \varepsilon\Omega\eta(q-Q)]^{-1} = I - \varepsilon\Omega\eta(q-Q)[I + \varepsilon\Omega\eta(q-Q)]^{-1} = I - \varepsilon[I + \varepsilon\Omega\eta(q-Q)]^{-1}\Omega\eta(q-Q).$$

On substituting K as given by (8.17) into the right side of (8.16) we obtain

$$(8.20) \qquad K = -\int_{q_0}^{q_1} \delta(q-Q)\Omega\,\eta(q-Q)dq + \varepsilon\int_{q_0}^{q_1} \delta(q-Q)$$

$$\cdot \left\{\int_{q_0}^{q_1} \delta(q'-Q)\left[I + \varepsilon\Omega\eta(q'-Q)\right]^{-1}\Omega\,\eta(q'-Q)dq'\right\} \eta(q-Q)\Omega\,\eta(q-Q)dq$$

$$= -\int_{q_0}^{q_1} \delta(q-Q)\Omega\,\eta(q-Q)dq$$

$$+ \varepsilon\int_{q_0}^{q_1} \delta(q-Q)\left[I + \varepsilon\Omega\eta(q-Q)\right]^{-1}\Omega\,\eta(q-Q)\,\eta(q-Q)\Omega\,\eta(q-Q)dq$$

$$= -\int_{q_0}^{q_1} \delta(q-Q)\Omega\eta(q-Q)dq + \varepsilon\int_{q_0}^{q_1} \delta(q-Q)\left[I + \varepsilon\Omega\eta(q-Q)\right]^{-1}\Omega\eta(q-Q)\Omega\eta(q-Q)dq.$$

In deriving equation (8.20) we have used the fact that $\delta(q-Q)\delta(q'-Q) = \delta(q-Q)\delta(q'-q)$.

On using (8.19) we have

$$(8.21) \qquad K = -\int_{q_0}^{q_1} \delta(q-Q)\Omega\,\eta(q-Q)dq + \int_{q_0}^{q_1} \delta(q-Q)\Omega\eta(q-Q)dq$$

$$-\int_{q_0}^{q_1} \delta(q-Q)\ \left[I - \varepsilon\Omega\eta(q-Q)\right]^{-1}\Omega\eta(q-Q)dq\,,$$

which is just (8.17). Hence (8.17) satisfies (8.16).

Now the operator $\left[I + \varepsilon\Omega\eta(q-Q)\right]^{-1}$ will always exist (and hence (8.17) will provide a solution for K) if Ω is bounded less than $\frac{1}{|\varepsilon|}$, for in this case $\Omega\eta(q-Q)$ is also bounded less than $\frac{1}{|\varepsilon|}$ (the bound of $\eta(q-Q)$ being less than 1). Then $\left[I + \varepsilon\Omega\eta(q-Q)\right]^{-1}$ is given by the series

$$(8.22) \qquad \left[I + \varepsilon\Omega\eta(q-Q)\right]^{-1} = \sum_{n=0}^{\infty} \varepsilon^n(-1)^n\left[\Omega\eta(q-Q)\right]^n .$$

Substitution of (8.22) into (8.17) gives the result for K which one would obtain from iteration.

Incidentally, the condition that Ω is bounded less than $\frac{1}{|\varepsilon|}$ assures us that $W = I + \varepsilon\,\Omega$ is positive definite.

Having obtained K we can find U, on using (8.19), to be

$$(8.23)\qquad U = I + \varepsilon K = I - \varepsilon \int_{q_0}^{q_1} \delta(q-Q)\left[I + \varepsilon\Omega\,\eta(q-Q)\right]^{-1}\Omega\,\eta(q-Q)\,dq$$

$$= \int_{q_0}^{q_1} \delta(q-Q)\ \left[I + \varepsilon\Omega\,\eta(q-Q)\right]^{-1}\,dq$$

and U_o to be

$$(8.24)\qquad U_o = WU^{*} = (I + \varepsilon\Omega)\int_{q_0}^{q_1} \left[I + \varepsilon\ \eta(q-Q)\Omega\right]^{-1}\delta(q-Q)\,dq$$

$$= \int_{q_0}^{q_1} \left\{\left[I + \varepsilon\ \eta(q-Q)\Omega\right] + \varepsilon\ \eta(Q-q)\Omega\right\}\left[I + \varepsilon\,\eta(q-Q)\Omega\right]^{-1}\delta(q-Q)\,dq$$

$$= I + \varepsilon\int_{q_0}^{q_1} \eta(Q-q)\Omega\left[I + \varepsilon\ \eta(q-Q)\Omega\right]^{-1}\ \delta(q-Q)\,dq.$$

Hence K_o is given by

$$(8.25)\qquad K_o = \frac{1}{\varepsilon}\left[U_o - I\right] = \int_{q_0}^{q_1} \eta(Q-q)\Omega\left[I + \varepsilon\,\eta(q-Q)\Omega\right]^{-1}\delta(q-Q)\,dq.$$

An alternative expression for K_o will prove useful later. It arises from the identity

$$(8.26)\qquad \Omega\left[I + \varepsilon\ \eta(q-Q)\Omega\right]^{-1} = \left[I + \varepsilon\Omega\,\eta(q-Q)\right]^{-1}\Omega.$$

The identity (8.26) can be proved by multiplying on the right by $\left[I + \varepsilon\ \eta(q-Q)\Omega\right]$ and on the left by $\left[I + \varepsilon\Omega\,\eta(q-Q)\right]$. The alternative form K_o is then

$$K_o = \int_{q_o}^{q_1} \eta(Q-q)\left[I + \varepsilon\Omega\eta(q-Q)\right]^{-1}\Omega\,\delta(q-Q)dq. \tag{8.27}$$

On comparing (8.27) with (8.13) it is clear that K_o has the desired triangularity properties.

It is easily verified that K_o and K as given by (8.27) and (8.17) satisfy (8.3) as required.

9. Uniqueness Theorems

In this section we prove the uniqueness of the solutions (8.17) and (8.27) for K and K_o. The proof presented here depends on the positive definite character W.

Theorem. If there is a solution U, U_o of $WU^* = U_o$ such that

$$< q|U|q' > = \delta(q-q') + \varepsilon < q|K|q' >$$

and

$$< q|U_o|q' > = \delta(q-q') + \varepsilon < q|K_o|q' >,$$

where $< q|K|q' >$ and $< q|K_o|q' >$ are bounded functions of q and q' and are triangular in the sense that they vanish when $q' > q$, then this solution is the only solution of this character.

Proof. Assume that there is another such solution $U^{(1)} = I + K^{(1)}$ and $U_o^{(1)} = I + K_o^{(1)}$. Then in addition to having the relation

$$WU^* = U_o \tag{9.1}$$

we should have

$$WU^{*(1)} = U_o^{(1)} . \tag{9.2}$$

On subtracting we find

$$W(K^{*}-K^{*(1)}) = (K_o-K_o^{(1)}) \; . \tag{9.3}$$

Let us define R and R_o by

$$R = K - K^{(1)} \tag{9.4}$$

$$R_o = K_o - K_o^{(1)} \; .$$

From the properties of K, $K^{(1)}$, K_o, $K_o^{(1)}$ we find that $< q|R|q' >$ and $< q|R_o|q' >$ vanish for $q' > q$ and that they are bounded functions of q and q'. Equation (9.3) can be written as

$$\int_{q_o}^{q_1} < q|W|q'' > dq'' < q''|R^{*}|q' > = < q|R_o|q' > , \tag{9.5}$$

from which

$$\int_{q_o}^{r} dq \int_{q_o}^{q'} dq'' < r|R|q > < q|W|q'' > < q''|R^{*}|q' > \tag{9.6}$$

$$= \int_{q'}^{r} < r|R|q > dq < q|R_o|q' ,$$

where $r > q'$. Let us now let r approach q'. Under the boundedness conditions assumed for R and R_o, the integral on the right of (9.6) vanishes. Hence

$$\int_{q_o}^{q'} dq \int_{q_o}^{q'} dq'' < q'|R|q > < q|W|q'' > < q''|R^{*}|q' > = 0. \tag{9.7}$$

If we regard $< q|R^{*}|q' >$ as a vector, say $|\Phi >$, insofar as it is a function of q, and if we regard q' as a fixed parameter, then equation (9.7) has the form

$$< \Phi |W| \Phi > = 0 \; . \tag{9.8}$$

Since W is positive definite, it follows that $|\Phi > = 0$, or, equivalently $R = 0$. Hence $U = U^{(1)}$, $U_o = U_o^{(1)}$ and the solution is unique.

In Appendix IV we present an alternative proof of the uniqueness. However, it is not quite so useful as the present proof, because instead of using the positive definite character of W, it assumes that $\varepsilon \Omega = W - I$ has an inverse.

10. Proof that $U^{-1} = U_o$

Thus far we have discussed only equation

$$(10.1) \qquad WU^* = U_o$$

and have shown that under quite general conditions, this equation has a unique solution if we specify that $\varepsilon K_o = U_o - I$ and $\varepsilon K = U - I$ are to be triangular in a specified representation. We have not yet discussed our other requirement on U and U_o, namely

$$(10.2) \qquad U^{-1} = U_o .$$

It might be expected that the solutions U and U_o which are obtained from (10.1) using the triangularity conditions of K and K_o would satisfy (10.2) only under rather exceptional conditions on the weight operator W, since equations (10.1) and (10.2) appear to be quite independent of one another. Surprisingly, however, it turns out that rather general conditions on W are sufficient to insure (10.2).

We give two proofs that $U^{-1} = U_o$. One is an abstract proof (using only the equation (10.1)) and the other makes use of the explicit solutions (8.23) and (8.24) for U and U_o. We present the abstract proof here because it is valid under rather more general conditions. The second proof is given in Appendix V.

Theorem. If $< q|K|q' >$ and $< q|K_o|q' >$ vanish when $q' > q$ and are bounded continuous functions of q and q' for $q < q'$, then $U^{-1} = U_o$.

Proof. We shall prove this theorem in two parts. First we shall prove $UU_o = I$ (i.e., U_o is the right inverse of U); then we shall show that U^{-1} exists, from which it follows that $U_o = U^{-1}$.

Let us first prove $UU_0 = I$ for the case where $< q|K|q' >$ and $< q|K_0|q' >$ are matrices (i.e., where Q has a discrete spectrum)[†]. Our requirements on K and K_0 are then

$$< q|K|q' > = 0, \qquad q' \geq q, \tag{10.3}$$

$$< q|K_0|q'> = 0, \qquad q' \geq q.$$

Now from (10.1) we have

$$UW = U_0^* \tag{10.4}$$

or

$$UWU^* = U_0^*U^* = UU_0. \tag{10.5}$$

The second equation follows from (10.1). From (10.5) we have, on using $U = I + \varepsilon K$, $U_0 = I + \varepsilon K_0$,

$$K + K_0 + \varepsilon KK_0 = K^* + K_0^* + \varepsilon K_0^* K^*, \tag{10.6}$$

or, in terms of the Q-representation,

$$\begin{aligned} &< q|K|q' > + < q|K_0|q' > + \varepsilon < q|KK_0|q' > \\ &\qquad = < q|K^*|q' > + < q|K_0^*|q' > + \varepsilon < q|K_0^*K^*|q' >. \end{aligned} \tag{10.7}$$

Now from (10.3) it can be shown that $< q|KK_0|q' > = 0$ for $q' \geq q$. Hence the right-hand side of (10.7) vanishes when $q' \geq q$. But since

$$< q|K^*|q' > = < q|K_0^*|q' > = < q|K_0^*K^*|q'> = 0, \quad q' \leq q,$$

it follows that the left-hand side vanishes when $q' \leq q$. Hence the right and left members of (10.7) vanish identically, i.e.,

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[†] This proof was suggested to the authors by Professors J. B. Keller and B. Friedman.

$$K + K_o + \varepsilon KK_o = 0 = K^* + K_o^* + \varepsilon K_o^* K. \tag{10.8}$$

Hence

$$UU_o = I + \varepsilon K + \varepsilon K_o + \varepsilon KK_o = I, \tag{10.9}$$

as required.

We shall now indicate the generalization of this proof to the case where Q has a continuous spectrum. The sole difference is that we do not want or need to require that $< q|K|q > = < q|K_o|q > = 0$, that is, that K and K_o be zero on the diagonal. In this case our previous considerations lead us to the conclusion that

$$< q|K|q' > + < q|K_o|q' > + \varepsilon < q|KK_o|q' > = 0 \tag{10.10}$$

for $q < q'$ or $q > q'$,

and

$$< q|K|q > + < q|K_o|q > + \varepsilon < q|KK_o|q > \neq 0 \tag{10.10a}$$

generally.

This expression (10.10a) will under our conditions be finite, however, for all q. Let $K + K_o + \varepsilon + KK_o$ operate on a state $|\emptyset >$ in Hilbert space whose Q-representative $< q|\emptyset >$ is a continuous function of q. Then the Q-representative of $(K + K_o + \varepsilon KK_o)|\emptyset >$ is given by

$$\int_{q_o}^{q} \left[< q|K|q' > + < q|K_o|q' > + \varepsilon < q|KK_o|q' >\right] dq' < q'|\emptyset >.$$

Clearly this integral equals zero, since, for fixed q, the integrand is zero for all q' except possibly for q' = q, where it is finite. Hence

$$(K + K_o + \varepsilon KK_o)\ |\emptyset > = 0 \tag{10.11}$$

for that set of states $|\emptyset >$ whose Q-representatives $< q|\emptyset >$ are continuous. Since this set of states is dense in Hilbert space, one can extend the definition of $K + K_o + \varepsilon KK_o$ to the whole Hilbert space, in which case (10.11) holds for all states $|\emptyset>$ in the Hilbert space. Hence, by definition, (10.8) holds, and, finally, so does (10.9).

We proceed now to the second part of the proof. We shall show that under the conditions of the theorem U has an inverse. The inverse $U^{-1} = (I+\varepsilon K)^{-1}$ obviously exists if the bound of K is less than $1/|\varepsilon|$. However, this condition is too restrictive and does not make use of the triangularity properties of K. A necessary and sufficient condition for U to have on inverse is that the equation $U|\emptyset> = 0$, when applied to any state $|\emptyset>$ in Hilbert space, imply that $|\emptyset> = 0$.

We shall now appeal to the theory of the Volterra integral equation to show that U satisfies this condition if U acts on the set of states $|\emptyset>$ whose Q-representatives $<q|\emptyset$ are continuous functions of $<q|\emptyset>$. $U|\emptyset> 0$ can be written as

$$(10.12) \qquad <q|\emptyset> + \varepsilon \int_{q_o}^{q} <q|K|q'> dq' <q'|\emptyset> = 0.$$

According to the Volterra theory [7] the only continuous function $<q|\emptyset>$ which can satisfy this equation if $<q|K|q'>$ is a continuous, bounded function of q and q' is $<q|\emptyset> = 0$.

Hence U^{-1} exists when applied to the set of states $|\emptyset>$ which have continuous Q-representatives. It follows from (10.4) that $U_oU = I$ when applied to any state in this set. Since the set of continuous representatives is dense in the whole Hilbert space, the extension of U_oU into the rest of Hilbert space must also be I. Finally, therefore,

$$(10.13) \qquad U_oU = I$$

over the whole Hilbert space.

In the case where q_o equals $-\infty$, our reasoning has to be modified slightly, for the Volterra theory cannot be applied directly to this case. Instead of considering all states whose representatives $< q|\emptyset >$ are continuous, we restrict ourselves to those continuous states for which a q_{min} exists such that $< q|\emptyset > = 0$ when $q < q_{min}$. The value of q_{min} may be arbitrarily small so long as it is finite. It may also be different for different states. Hence the equation (10.12) is replaced by

$$(10.12a) \qquad < q|\emptyset > + \int_{q_{min}}^{q} < q|K|q' > dq' < q'|\emptyset > = 0.$$

Then the Volterra theory may be applied to show that the only solution in this set is $< q|\emptyset > = 0$, and a reasoning similar to that used earlier shows that $U_oU = I$ when U_oU is applied to any state in this set. Since this set is dense in the whole Hilbert space, $U_oU = I$ in the whole Hilbert space as before.

The condition that $< q|K|q' >$ be continuous is a little more severe than we should like; we should prefer the condition that $< q|K|q' >$ be bounded, so that we might include the case where there are a finite number of point discontinuities. The proof in Appendix V is valid for such operators, provided we assume that (8.23) and (8.24) give the solutions for U and U_o and that the operators W^{-1} and Ω^{-1} exist.

11. The Second Asymptotic Form for the Eigenfunctions of H: The Weight Operator as an Analogue to the Scattering Operator

We have seen that the triangularity properties of K and K_o in the Q-representation imply that $< q|H,H;E,a >$ asymptotically approaches the eigenfunction of H_o, $< q|H_o,A_o;E,a >$, as q approaches q_o (cf. (7.1) through (7.4)). One might ask for the asymptotic form of $< q|H,A;E,a >$ as q approaches q_1. We shall now show that then $< q|H,A;E,a >$ approaches another eigenfunction of H_o.

Let us define the state $|H_o,B_o;E,a>$ by

(11.1) $$|H_o,B_o;E,a> = W^{-1}|H_o,A_o;E,a>.$$

First we show that W^{-1} commutes with H_o and that as a consequence $|H_o,B_o;E,a>$ is an eigenfunction of H_o.

Clearly, since W commutes with H_o, we have

(11.2) $$H_o = WH_oW^{-1}.$$

On multiplying on the left by W^{-1} we find that

(11.3) $$W^{-1}H_o = H_oW^{-1},$$

i.e., that W^{-1} commutes with H_o. Now

(11.4) $$\begin{aligned} H_o|H_o,B_o;E,a> &= H_o\ W^{-1}|H_o,A_o;E,a> = W^{-1}\ H_o|H_o,A_o;E,a> \\ &= E\ W^{-1}|H_o,A_o;E,a> \\ &= E|H_o,B_o;E,a> . \end{aligned}$$

Hence $|H_o,B_o;E,a>$ is an eigenstate of H_o corresponding to the eigenvalue E.

There are obviously as many eigenstates $|H_o,B_o;E,a>$ for a fixed eigenvalue of H_o as there are eigenstates $|H_o,A_o;E,a>$. Hence the degeneracy label a in $|H_o,B_o;E,a>$ has the same range as the label a in $|H_o,A_o;E,a>$. Since generally $|H_o,B_o;E,a>$ is not an eigenstate of A_o (this follows from the fact that A_o does not usually commute with W^{-1}), we have introduced the operator B_o' as the operator whose eigenstate $|H_o,B_o;E,a>$ corresponds to the eigenvalue a.

Equation (11.1) shows that the completeness and orthonormality relations for $|H_o,B_o;E,a>$ involve the use of a weight operator W^2. They are

(11.5) $$< H_o,B_o;E,a\ |W^2|H_o,B_o;E',a' > = \delta(E-E')\delta(a,a'),$$

(11.6) $$\iint |H_o,B_o;E,a > dE\ da < H_o,B_o;E,a\ |W^2 = I.$$

We maintain that if q_1 is finite,

$$(11.7) \qquad \lim_{q \to q_1} \left[< q|H,A;E,a > - < q|H_o,B_o;E,a > \right] = 0,$$

i.e. $< q|H,A;E,a >$ approaches the eigenfunction $< q|H_o,B_o;E,a >$ of H_o asymptotically as q approaches q_1. To prove this we note that

$$(11.8) \qquad W = U^{-1} U^{*-1} = U_o U_o^*$$

(cf. (6.8). But

$$(11.9) \qquad \begin{aligned} |H,A;E,a > &= U|H_o,A_o;E,a > \\ &= U\, W|H_o,B_o;E,a > \\ &= U_o^{-1} W|H_o,B_o,E,a > \\ &= U_o^*|H_o,B_o;E,a > . \end{aligned}$$

Hence, since the triangularity condition on K_o,

$$(11.10) \qquad < q|K_o|q' > = 0 \quad \text{for} \quad q' > q \ ,$$

implies

$$(11.11) \qquad < q|K_o^*|q' > = 0 \ \text{ for } q' < q,$$

equation (11.9) can be written

$$(11.12) \qquad \begin{aligned} &< q|H,A;E,a > \\ &= < q|H_o,B_o;E,a > + \int_q^{q_1} < q|K_o^*|q' > dq' < q'|H_o,B_o;E,a > , \end{aligned}$$

from which (11.7) follows, provided q_1 is finite.

If $q_1 = +\infty$, the kernel $< q|K_o^*|q' >$ must satisfy some additional conditions in order that $\lim_{q \to +\infty} \int_q^{\infty} < q|K_o^*|q' > dq' < q'|H_o,B_o;E,a >$ vanish. In the case of the one-dimensional scattering problem ($-\infty < x < +\infty$) these extra conditions are generally not satisfied; hence (11.7) is not satisfied (see ref. [4]).

In the boundary-value problem of finding the eigenfunctions $|H,A;E,a>$ the operator W^{-1} plays a role very similar to that of the scattering operator S in the initial-value problem. In the initial-value problem if one prescribes that at $t = -\infty$ the solution of the time-dependent Schrodinger equation be $|H_o,A_o;E,a>$, then at $t = +\infty$ the solution of the time-dependent Schrodinger equation is given by $|H_o,C_o;E,a>$ where

$$(11.13) \qquad |H_o,C_o;E,a> = S|H_o,A_o;E,a>.$$

It can be shown that (cf. Eq. (4.10)) as a consequence of the fact that the scattering operator commutes with H_o, $|H_o,C_o;E,a>$ is also an eigenfunction of H_o. That is, the scattering operator S relates the two asymptotic states of the time-dependent Schrodinger equation, which are also eigenstates of H_o.

Similarly, from equation (11.1) the operator W^{-1} relates the asymptotic forms in space of the eigenfunction $|H,A;E,a>$. The operator W^{-1} differs essentially from S however, in that it is a Hermitian rather than a unitary operator. In the time-dependent case if the eigenfunctions $|H_o,A_o;E,a>$ are normalized to a δ-function, so are the eigenfunctions $S|H_o,A_o;E,a>$; however, as mentioned above, the eigenfunctions $W^{-1}|H_o,A_o;E,a>$ will not have a δ-function normalization.

12. Generalizations

Up to now we have assumed that Q by itself forms a complete set of commuting variables and looked for a triangular K in terms of this representation. Let us now assume that $Q_1, Q_2, \cdots Q_n$ form a complete set of commuting variables. Then K and K_o, in terms of this representation, are given by

$$< q_1, q_2, \cdots q_n | K | q_1', q_2', \cdots q_n' >$$

and

$$< q_1, q_2, \cdots q_n | K_o | q_1', q_2', \cdots q_n' > .$$

If we assume that K and K_o are triangular in only one of the variables, q_1, say, i.e., if

$$< q_1, q_2, \cdots q_n | K | q_1', q_2', \cdots q_n' > = 0, \qquad q_1' > q_1$$

$$< q_1, q_2, \cdots q_n | K_o | q_1', q_2', \cdots q_n' > = 0, \qquad q_1' > q_1$$

and if these operators obey the appropriate boundedness and continuity conditions in terms of the variables q_1 and q_1', the previous theory applies without any essential alteration.

In fact, if one writes operators, say P, and vectors $|\emptyset>$ in the $Q_1 \cdots Q_n$ representation as

$$< q_1, q_2, \cdots q_n | P | q_1', q_2', \cdots q_n' > \equiv < q_1 | P | q_1' >$$

and

$$< q_1, q_2, \cdots q_n | \emptyset > \equiv < q_1 | \emptyset > ,$$

and regards $< q_1 | P | q_1' >$ as an operator and $< q_1 | \emptyset >$ as a vector in terms of the remaining variables, one can go through the previous discussion and see that the formulas remain unaltered. One need merely interpret continuity and boundedness in a somewhat more general fashion to preserve the various theorems. We refrain from doing so explicitly, since the procedure is almost obvious.

The following much greater generalization appears possible. If we require that K and K_o have the same invariant subspaces in Hilbert space, and that K^* and K_o^* have invariant subspaces orthogonal to those of K, K_o, the theory appears to be valid. If one works in a proper representation one can generalize the concept of triangularity in terms of a suitable Q-representation and carry out the previous procedure. Perhaps the writers will sketch this method in later papers.

13. Relation to the Wiener-Hopf Integral Equation

The Wiener-Hopf integral equation for two unknown functions f(z) and g(z) is written

$$f(z) = g(z) + \gamma(z) + \int_{-\infty}^{z} f(z-\tau)\,\beta(\tau)d\tau . \tag{13.1}$$

Here z ranges from $-\infty$ to $+\infty$. The functions $\gamma(z)$ and $\beta(z)$ are given, and f(z) and g(z), the unknown functions, satisfy the boundary conditions

$$\begin{aligned} f(z) &= 0 \qquad z < 0 \\ g(z) &= 0 \qquad z > 0 . \end{aligned} \tag{13.2}$$

It has been shown (e.g., ref. [8]) that with certain restrictions on $\gamma(z)$ and $\beta(z)$ one may solve for both unknown functions explicitly. The usual procedure makes use of the analytic properties of these functions. Now we shall show that our equations for K and K_o reduce in special cases to the standard Wiener-Hopf equation (13.1). In terms of the Q-representation, we have (cf. (8.4))

$$\begin{aligned} < q|K|q' > &= < q|K_o^*|q' > - < q|\Omega|q' > \\ &\quad - \varepsilon \int_{q_o}^{q} < q|K|q'' > dq'' < q'' |\Omega|q' >. \end{aligned} \tag{13.3}$$

This is an equation for both K and K_o^*. Let us now take q_o and q_1 as being $-\infty$ and $+\infty$ respectively. Furthermore, let us assume $< q|\Omega|q' >$, $< q|K|q' >$ and $< q|K_o^*|q' >$ are functions of the difference $q-q'$ only. We shall write

$$< q|K|q' > = K(q-q')$$

$$< q|K_o^*|q'> = K_o^*(q-q')$$

$$< q|\Omega|q' > = \Omega(q-q')$$

where, in accordance with the triangularity conditions,

$$(13.4) \qquad K(q-q') = 0 \qquad q' > q$$

$$K_o^*(q-q') = 0 \qquad q' < q.$$

Equation (13.3) is then

$$(13.5) \qquad K(q-q') = K_o^*(q-q') - \Omega(q-q') - \varepsilon \int_{-\infty}^{q} K(q-q'')\Omega(q''-q')dq'' .$$

If we now write

$$q-q' = z$$

$$q''-q' = \tau ,$$

equation (13.5) becomes

$$(13.5a) \qquad K(z) = K_o^*(z) - \Omega(z) - \varepsilon \int_{-\infty}^{z} K(z-\tau)\Omega(\tau)d\tau,$$

where in accordance with (13.4)

$$K(z) = 0 \qquad z < 0$$

$$K_o^*(z) = 0 \qquad z > 0.$$

Identifying $K(z)$ with $f(z)$, $K_o^*(z)$ with $g(z)$, $-\Omega(z)$ with $\gamma(z)$, and $-\varepsilon\Omega(z)$ with $\beta(z)$, we see that (13.5) is of the form (13.1). In a sense, then, equation (13.3) is a generalization of the Wiener-Hopf equation. In another sense, however, it is more special, since for our applications we insist that Ω be Hermitian and have other appropriate properties.

Our procedure bears another resemblance to the Wiener-Hopf techniques, that is, to the factorization process. In the Wiener-Hopf approach, one factors an operator analogous to W into a product of two exponentials. With our formalism we are able to carry out this factorization explicitly.

From

$$WU^* = U_o \tag{13.6}$$

and

$$U_o = U^{-1} \tag{13.7}$$

it is easy to show

$$W = U_o U_o^*. \tag{13.8}$$

We can write

$$W = e^A e^{A^*}. \tag{13.9}$$

If K_o is triangular in the Q-representation, we have

$$A = \log U_o = \log \left\{ I + \varepsilon \int_{q_o}^{q_1} \eta(Q-q) \left[I + \varepsilon \Omega \eta(q-Q) \right]^{-1} \Omega \, \delta(q-Q) dq \right\}, \tag{13.10}$$

and hence A also is triangular in this representation. From (13.7) we may write

$$U = U_o^{-1} = e^{-A} . \tag{13.11}$$

Equations (13.9) and (13.11) suggest the following alternative procedure for obtaining triangular solutions of (13.6) which satisfy (13.7). Since W is positive definite, log W can be defined. Equation (13.9) can then be written

$$e^{\log W} = e^A e^{A^*}. \tag{13.12}$$

If we can find an operator A which is triangular in the Q-representation such that (13.12) holds, then it can be shown that a solution of (13.6) which satisfies both the triangularity requirement and equation (13.7) is

$$U_o = e^A, \qquad U = e^{-A} . \tag{13.13}$$

14. The Case where the Spectra of H and H_o do not Coincide

From our theory the operators U and U_o can provide a Hamiltonian H, where

$$H = UH_oU_o = UH_oU^{-1} \tag{14.1}$$

such that the spectrum of H coincides with that of H_o. This fact follows from the result that H and H_o are related by a canonical transformation.

If we wish to obtain Hamiltonians H only part of whose spectrum coincides with that of H_o, and if we wish to preserve at least an analogue of (14.1) and the equation

$$WU^* = U_o, \qquad U_o = U^{-1}, \tag{14.2}$$

then we must extend the domain of definition of H_o and H such that the spectra of the extended operators H_o and H coincide and consist of the sum of that part of the spectrum of H which coincides with the spectrum of H_o, that part of the spectrum of H which lies outisde of the spectrum of H_o, and that part of the spectrum of H_o which lies outside the spectrum of H. Then for the extended operators H and H_o it is possible that operators U, $U_o = U^{-1}$ exist such that

$$\tilde{H} = U\tilde{H}_oU^{-1} . \tag{14.3}$$

Two facts ought to be noted about the extension: 1) the extension can not be completely arbitrary because the perturbation $\varepsilon V = \tilde{H} - \tilde{H}_o$, in order to be useful, must also be equal to $\varepsilon V = H - H_o$. For only in this case can it be said that $H = H_o + \varepsilon V$ has a meaning; 2) the space of the extended operators $\tilde{H}$ and $\tilde{H}_o$ may be greater than Hilbert space. This can be seen from the special examples treated in reference [2], for example, where the Hamiltonian H has point eigenvalues in addition to a continuous spectrum which coincides with that of H_o, and where H_o has only this continuous spectrum. "Eigenfunctions" of H_o are introduced whose "eigenvalues" equal the point eigenvalues of H.

These " eigenfunctions" of H_o are not those which span Hilbert space. Their introduction corresponds to an extension of the definition of H_o into a larger space. We hope to indicate the nature of the extension for general cases in a more abstract fashion in a later paper.

15. A Necessary Condition for the Triangularity of K, K_o

For many weight operators W there are generally many representations such that we can find unique operators $U = I + \varepsilon K$ and $U_o = I + \varepsilon K_o$ such that K and K_o are triangular in these representations. To get unique solutions for U and U_o, one must specify which of these representations one wishes to use; one should choose from physical considerations the one in which the perturbed eigenfunctions approach the unperturbed eigenfunctions asymptotically.

It should not be assumed, however, that once a positive definite operator $W = I + \varepsilon \Omega$ has been given it is possible to choose any representation for triangularity of K, K_o. Generally, once W has been given, there is a restricted class of representations in which K, K_o can be triangular.

A useful necessary condition, which we have discussed earlier, for the triangularity of K and K_o is the following: In order that K, K_o be triangular in the Q-representation, the operator

$$\int_{q_0}^{q_1} \delta(q-Q)\Omega\,\eta(q-Q)dq$$

must exist.

This condition, which follows from equation (8.16), enables us to discard representations in which K, K_o cannot be triangular. For example, if Ω is a constant, then the operator

$$\int_{q_0}^{q_1} \delta(q-Q)\Omega\,\eta(q-Q)dq = \Omega \int_{q_0}^{q} \delta(q-Q)\,\eta(q-Q)dq.$$

But the operator $\int_{q_0}^{q} \delta(q-Q)\,\eta(q-Q)dq$ is not defined for any Q-representation. Hence if Ω, or equivalently W, is a constant, K, K_o cannot be triangular in any representation. At most we can require that K, K_o be diagonal in the Q-representation, i.e., singular along the diagonal. In this case it can be shown that the most general solution for U and U_o which satisfies $WU^* = U_o$, $U_o = U^{-1}$ is

$$(15.1) \qquad \begin{aligned} U &= (W)^{-\frac{1}{2}} e^{i\theta(Q)}, \\ U_o &= (W)^{\frac{1}{2}} e^{-i\theta}(Q), \end{aligned}$$

where $\theta(Q)$ is an arbitrary real function of Q. In the case, for example, where H_o in the Q-representation is given by $-i\frac{\partial}{\partial q}$ $(-\infty < q < \infty)$, we have

$$(15.2) \qquad H = -i\frac{\partial}{\partial q} + \theta'(q),$$

where the prime indicates the derivative.

An example of a representation in which K, K_o cannot be triangular for any Ω is the H_o-representation, for in this case, since Ω always commutes with H_o, we have

$$(15.3) \qquad \int_{q_0}^{q_1} \delta(q-H_o)\Omega\,\eta(q-H_o)dq = \Omega \int_{q_0}^{q_1} \delta(q-H_o)\,\eta(q-H_o)dq$$

which again is undefined. In (15.3) we have labeled the eigenvalues of H_o by q instead of E, as in earlier sections of the paper. As before, the requirement nearest to triangularity which we can impose is that K, K_o be diagonal in the H_o representation In this case the general solution for U, U_o is similar to that obtained for the previous case, namely

(15.4)
$$U = (W)^{-\frac{1}{2}} e^{i\theta(H_o)},$$
$$U_o = (W)^{\frac{1}{2}} e^{-i\theta(H_o)},$$

where $\theta(H_o)$ is an arbitrary real function of H_o. Now we have

(15.5)
$$H = H_o$$

and the effect of the transformation U is merely to change the normalization of the eigenfunctions of H_o.

16. A Simple Example

An example which provides us with exact, unique solutions U, U_o of $WU^* = U_o$ such that $U = U_o^{-1}$ and K, K_o are triangular in a specific representation will now be given. Let us take the Q-representation to be the usual x-representation, i.e., let the eigenvalues q lie in a continuum between $-\infty$ and $+\infty$. Further, let H_o be

(16.1)
$$H_o^Q = -i \frac{d}{dq},$$

where the superscript Q denotes that H_o is expressed in the Q-representation. Then it is clear the spectrum of H_o lies in the continuum between $-\infty$ and $+\infty$. The eigenfunctions in the Q-representation are given by

(16.2)
$$< q|H_o;E > = \frac{1}{\sqrt{2\pi}} e^{iEq}.$$

In the previous section we have already considered the case where W is a constant. Let us choose an Ω such that $W = I + \varepsilon \Omega$ is positive definite and $\int \delta(q-Q)\Omega\,\eta(q-Q)dq$ exists. Such an Ω is given by

(16.3) $$\Omega = \frac{2\alpha}{H_o^2 + \alpha^2}, \qquad \alpha > 0$$

and

(16.4) $$W = I + 2\varepsilon \frac{\alpha}{H_o^2 + \alpha^2}.$$

It is clear that W is positive definite for $\varepsilon \geq -\frac{\alpha}{2}$. We shall restrict ourselves to such values of ε. It is easy to show that

(16.5) $$< q|\Omega|q' > = 2\alpha \int_{-\infty}^{+\infty} < q|H_o;E > \frac{1}{E^2+\alpha^2} < H_o;E|q' > dE = e^{-\alpha|q-q'|}.$$

To show that $\int \delta(q-Q)\Omega\eta(q-Q)dq$ exists we write it in the Q-representation:

(16.6) $$< q|\int \delta(q''-Q)\Omega\, \eta(q''-Q)|q' >$$

$$= < q|\Omega|q' > \eta(q-q')$$

$$= e^{-\alpha(q-q')}\eta(q-q').$$

The integral operator with this kernel clearly exists. The integral equation for $< q|K|q' >$ is

(16.7) $$< q|K|q' > = -e^{-\alpha(q-q')} - \varepsilon \int_{-\infty}^{q} < q|K|q'' > e^{-\alpha|q''-q'|} dq''.$$

It is easy to show that a solution of this equation is

(16.8) $$< q|K|q' > = \frac{1}{\varepsilon}(\alpha-\gamma)e^{-\gamma(q-q')}\eta(q-q'),$$

where

(16.8a) $$\gamma = \sqrt{\alpha^2 + 2\alpha\varepsilon},$$

or

(16.9) $$< q|U|q' > = \delta(q-q') + (\alpha-\gamma)e^{-\alpha(q-q')}\eta(q-q').$$

Hence

(16.11) $$< q|U_o|q' > = \delta(q-q') + (\gamma-\alpha)e^{-\alpha(q-q')}\eta(q-q').$$

K, K_o satisfy the conditions for the uniqueness theorem of Section 9.

Hence (16.9) and (16.11) are the only solutions of $WU^* = U_o$ in which K and K_o have the desired triangularity properties.

Actually, (16.7) has another solution for $\varepsilon < 0$. It is

$$< q|K|q' > = \frac{1}{\varepsilon} (\alpha+\gamma)e^{\gamma(q-q')} \eta(q-q'). \tag{16.12}$$

However, it can be shown that this solution has the property

$$\int_{-\infty}^{q'}\int_{-\infty}^{q'} < q'|K|q > dq < q|W|q'' > dq'' < q'' |K^*|q' > = \frac{1}{\varepsilon} \frac{(\alpha+\gamma)}{(\alpha-\gamma)} < 0 \tag{16.13}$$

since $\varepsilon < 0$, and $\alpha > \gamma$. Hence $< q|K^*|q' >$ considered as a function of q, with q' fixed, is outside the domain where W is positive definite. But since we prescribe W to be positive definite, we reject this solution. On the other hand, the first-mentioned solutions for K and K_o are in the domain where W is positive definite. This example illustrates how the positive definite character of W is used to select unique solutions for K.

It is easy to show that $U^{-1} = U_o$; using the explicit forms (16.9) and (16.11) and carrying out the integrations, the following is proved:

$$\int_{-\infty}^{+\infty} < q|U_o|q''> dq'' < q'' |U|q' > = \delta(q-q') \tag{16.14}$$

$$\int_{-\infty}^{+\infty} < q|U|q''> dq'' < q'' |U_o|q' > = \delta(q-q'). \tag{16.15}$$

Of course, this result was to be expected in view of the theorems of Section 10.

The example that we have discussed here has the virtue of illustrating the theorems discussed earlier. However, it is somewhat trivial in the sense that

$$H = UH_oU_o = H_o, \tag{16.16}$$

as follows from the fact that U commutes with H_o. In fact, with our choice of H_o and q, U will always commute with H_o for any W which is such that U, U_o can be obtained uniquely. This result is a consequence of the fact that $< q|U|q' >$ will always be a function of the difference q-q' and from this it follows that $< H_o;E|U|H_o;E' >$ is diagonal and thus commutes with H_o. We refrain from giving details. With these choices of W, however, the perturbation V = 0 is unique. Since the cases treated in previous papers are all special cases of the procedure discussed herein, they also provide examples of how to obtain the potential from the weight functions.

17. Triangularity of Operators and Asymptotic Conditions on Eigenfunctions when $q_o = -\infty$

In Section 7 it was indicated that triangularity of the operators K and K_o is not sufficient to guarantee that $< q|H,A;E,a >$ approaches $< q|H_o,A_o;E,a >$ as $q \to q_o$ in the case where $q_o = -\infty$ (or $+\infty$). The triangularity conditions are usually sufficient, however, to permit one to obtain unique solutions U, U_o, and hence a potential V, from a given weight operator W, as seen from the previous work.

In particular, the problem of Section 16 provides us with an example where the triangularity conditions yield unique operators U, U_o and V, but where the asymptotic conditions fail. To show how the conditions fail we note that on using K as obtained from Section 16 we have

$$(17.1) \qquad \int_{-\infty}^{q} < q|K|q' > dq' < q'|H_o;E > = \frac{(\alpha-\gamma)}{\varepsilon} (\gamma+iE)^{-1} e^{iEq}.$$

Hence the expression $\int_{-\infty}^{q} < q|K|q' > dq' < q'|H_o;E >$ clearly does not approach zero as $q \to -\infty$. We have in fact for finite q :

$$(17.2) \qquad < q|H;E > = < q|H_o;E > \left[1 + (\alpha-\gamma)(\gamma+iE)^{-1}\right].$$

Hence the eigenfunctions $< q|H;E >$ are also eigenfunctions of H_o, which, of course, was to be expected, since $H = H_o$.

We shall now give two sets of sufficient conditions on W (or equivalently Ω) to guarantee that the triangularity condition implies the asymptotic condition (7.1).

The first sufficient condition is based on the observation that if there is a q_{min} such that $< q|K|q' >$ vanishes when $q < q_{min}$, then

$$(17.3) \qquad \lim_{q \to -\infty} \int_{-\infty}^{q} < q|K|q' > dq' < q'|H_o,A_o;E,a > = 0 .$$

Theorem: If there is a q_{min} such that $< q|\Omega|q' >$ vanishes when $q < q_{min}$, then

$$\lim_{q \to -\infty} \left[< q|H,A;E,a > - < q|H_o,A_o;E,a >\right] = 0$$

Proof: Since Ω is Hermitian, the condition

$$(17.4) \qquad < q|\Omega|q' > = 0 \qquad\qquad q < q_{min}$$

implies

$$(17.5) \qquad < q|\Omega|q' > = 0 \qquad\qquad q' < q_{min} \quad .$$

The equation for $< q|K|q' >$ is

$$(17.6) \qquad < q|K|q' > = - < q|\Omega|q' > - \varepsilon \int_{q_{min}}^{q} < q|K|q'' > dq'' < q'' |\Omega|q' >, \; (q' < q).$$

Now when $q < q_{min}$ and $q' < q$,

$$(17.7) \qquad < q|K|q' > = 0,$$

as follows from (17.4), (17.5) and (17.6). Hence (17.3) is valid and the asymptotic condition holds.

The sufficient condition which we have stated is somewhat severe in that we may expect (17.3) to hold even if q_{min} does not exist, provided that $< q|\Omega|q' >$ approaches zero sufficiently rapidly as $q \rightarrow -\infty$. This leads us to our second sufficient condition.

Theorem: Let $< q|H_o,A_o;E,a >$ be a bounded function of q for sufficiently small q, and let $< q|K|q' >$ be an absolutely integrable function of q'; further, let $|< q|\Omega|q'>| \leq r(q')$ for all q, where r(q') is a function such that

$$\int_{-\infty}^{q} r(q')dq' < \infty.$$

Then

$$\lim_{q \rightarrow -\infty} \int_{-\infty}^{q} < q|K|q' > dq' < q'|H_o;A_o;E,a > = 0$$

and thus

$$\lim_{q \rightarrow -\infty} \Big[< q|H,A;E,a > - < q|H_o,A_o;E,a > \Big] = 0.$$

Proof: From (17.6) we have

$$|< q|K|q' >| \leq |< q|\Omega|q' >| + \int_{-\infty}^{q} |< q|K|q''>|dq'' |< q''|\Omega|q' >|, \tag{17.8}$$

where $q' < q$. However, from the conditions of the theorem and the Hermitian property of Ω, we see that if q (and hence q') is sufficiently small, then

$$|< q|\Omega|q' >| \leq r(q'), \tag{17.9}$$

$$|< q''|\Omega|q'>| \leq r(q'). \tag{17.10}$$

Hence

$$|< q|K|q' >| \leq r(q')+r(q') \int_{-\infty}^{q} |< q|K|q'' >|dq'' . \tag{17.11}$$

The integral on the right of (17.11) exists, in accordance with the assumptions of the theorem. For the same reason we can write

$$\int_{-\infty}^{q} |< q|K|q' >|dq' \leq \int_{-\infty}^{q} r(q')dq' \left[1 + \int_{-\infty}^{q} |< q|K|q''>|dq''\right], \tag{17.12}$$

or

$$\int_{-\infty}^{q} |< q|K|q' >|dq' \left[1 - \int_{-\infty}^{q} r(q')\right] \leq \int_{-\infty}^{q} r(q')dq' . \tag{17.13}$$

If q is sufficiently small we have

$$1 - \int_{-\infty}^{q} r(q')dq' > 0. \tag{17.14}$$

Hence for such values of q,

$$\int_{-\infty}^{q} |< q|K|q' >|dq' \leq \int_{-\infty}^{q} r(q')dq' \Big/ \left[1 - \int_{-\infty}^{q} r(q')dq'\right] . \tag{17.15}$$

Therefore

$$\lim_{q \to -\infty} \int_{-\infty}^{q} |< q|K|q' >|dq' = 0. \tag{17.16}$$

Now

$$\left| \int_{-\infty}^{q} < q|K|q' > dq' < q'|H_o,A_o;E,a > \right| \leq B \int_{-\infty}^{q} |< q|K|q' >|dq', \tag{17.17}$$

where B is the bound of $|<q|H_o,A_o;E,a >|$ which, in accordance with the assumptions of the theorem, must exist.

Hence, from (17.16) and (17.17)

$$\lim_{q \to -\infty} \int_{-\infty}^{q} < q|K|q' > < q'|H_o,A_o;E,a > = 0, \tag{17.18}$$

and our theorem is proved.

It should be noted that for the case discussed in Section 16 it is not possible to choose an operator Ω such that the above conditions are satisfied. This follows from the fact that $< q|\Omega|q' >$ is a function of the difference $q-q'$ for all suitable operators Ω which yield triangular operators K, K_o.

Appendix I

Expressions for $U_{\mp}$, S, S^{-1}

Using equation (3.7) we have

$$(\text{I-1}) \qquad e^{iH_ot}\, e^{-iHt}\, U_- = e^{iH_ot}\, U_-\, e^{-iH_ot}.$$

Using (3.9) with $U = U_-$, and using the fact that L commutes with H_o, we obtain

$$(\text{I-2}) \qquad e^{iH_ot}\, U_-\, e^{-iH_ot} = L + \varepsilon \int \frac{P}{E-H_o}\, e^{-i(E-H_o)t}\, VU_-\delta(E-H_o)dE.$$

Now it can be shown that

$$(\text{I-3}) \qquad \lim_{t \to \pm\infty} \frac{P}{E-H_o}\, e^{-i(E-H_o)t} = \mp\, i\pi\delta(E-H_o)$$

when both sides of the equation are applied to appropriate states in Hilbert space. Hence

$$(\text{I-4}) \quad \lim_{t \to -\infty} e^{iH_ot} e^{-iHt} U_- |\emptyset > = L|\emptyset > + i\pi \int \delta(E-H_o) VU_-\delta(E-H_o)dE|\emptyset > = |\emptyset >,$$

from which (4.5) follows. Equation (4.15) is obtained similarly. To get the expression for S one shows, using (I-3) and (4.5), that

$$(\text{I-5}) \quad \lim_{t \to +\infty} e^{iH_ot} e^{-iHt} U_- |\emptyset > = L|\emptyset > -i\pi \int \delta(E-H_o) VU_-\delta(E-H_o)dE|\emptyset >$$
$$= \left[I - 2i\pi \int \delta(E-H_o)\, VU_-\delta(E-H_o)dE \right] |\emptyset >.$$

The expression for S^{-1} is obtained similarly.

Appendix II

Proof of $U^*_\pm U_\pm = I$

Let us consider the equation with the negative subscript. From (4.6) we have

$$\text{(II-1)}\qquad U^*_-U_- = I + \varepsilon\int \gamma_-(E-H_o)VU_-\delta(E-H_o)dE + \varepsilon\int \delta(E-H_o)U^*_-V\gamma^*_-(E-H_o)dE$$

$$+ \varepsilon^2\iint \delta(E-H_o)U^*_-V\gamma^*_-(E-H_o)\gamma_-(E'-H_o)VU_-\delta(E'-H_o)dEdE',$$

where

$$\text{(II-2)}\qquad \gamma^*_-(E-H_o) = + i\pi\delta(E-H_o) + \frac{P}{E-H_o} = - \gamma_-(H_o-E) .$$

Now, it can be shown that the functions $\gamma_\pm(x)$ satisfy the identity

$$\text{(II-3)}\qquad \gamma_\pm(x)\gamma_\pm(y) = \gamma_\pm(x+y)\left[\gamma_\pm(x) + \gamma_\pm(y)\right].$$

Hence

$$\text{(II-4)}\qquad \gamma^*_-(E-H_o)\gamma_-(E'-H_o) = - \gamma_-(H_o-E)\gamma_-(E'-H_o)$$

$$= - \gamma_-(E'-E)\left[\gamma_-(H_o-E) + \gamma_-(E'-H_o)\right]$$

$$= \gamma_-(E'-E)\left[\gamma^*_-(E-H_o) - \gamma_-(E'-H_o)\right] .$$

Then the fourth term on the right of (II-1) is given by

$$\text{(II-5)}\qquad \varepsilon^2\iint \delta(E-H_o)U^*_-V\,\gamma^*_-(E-H_o)\gamma_-(E'-H_o)VU_-\delta(E'-H_o)dEdE'$$

$$= \varepsilon^2\iint \gamma_-(E'-E)\delta(E-H_o)U^*_-V\,\gamma^*_-(E-H_o)VU_-\delta(E'-H_o)dEdE'$$

$$- \varepsilon^2\iint \gamma_-(E'-E)\delta(E-H_o)U^*_-V\,\gamma_-(E'-H_o)VU_-\delta(E'-H_o)dEdE'$$

$$= \varepsilon^2\int \gamma_-(E'-H_o)\left[\int \delta(E-H_o)U^*_-V\,\gamma^*_-(E-H_o)dE\right]VU_-\delta(E'-H_o)dE'$$

$$- \varepsilon^2\int \delta(E-H_o)U^*_-V\left[\int \gamma_-(E'-H_o)VU_-\delta(E'-H_o)dE'\right]\gamma_-(H_o-E)dE.$$

In equation (II-5) we have used the fact that

$$\text{(II-6)}\qquad f(E)\delta(E-H_o) = f(H_o)\delta(E-H_o),$$

where f(E) is any function of E. In the last equation of (II-5), we write

$$\left[\varepsilon\int\delta(E-H_0)U_-^*V\,\gamma_-^*(E-H_0)dE\right] = U_-^* - I$$

$$\left[\varepsilon\int\gamma_-(E-H_0)VU_-\delta(E-H_0)dE\right] = U_- - I$$

so that after rearrangement of terms the fourth term on the right of (II-1) becomes

$$\text{(II-7)}\qquad \varepsilon^2\iint\delta(E-H_0)U_-^*V\,\gamma_-^*(E-H_0)\gamma_-(E'-H_0)VU_-\delta(E'-H_0)dEdE'$$

$$= -\varepsilon\int\gamma_-(E-H_0)VU_-\delta(E-H_0)dE - \varepsilon\int\delta(E-H_0)U_-^*V\,\gamma_-^*(E-H_0)dE$$

$$+\varepsilon\int\gamma_-(E-H_0)U_-^*VU_-\delta(E-H_0)dE - \varepsilon\int\delta(E-H_0)U_-^*VU_-\gamma_-(H_0-E)dE.$$

The first two terms on the right of (II-6) cancel the second and third terms on the right of (II-1).

We can write the last two terms on the right of (II-7) as

$$\text{(II-8)}\qquad \varepsilon\int\gamma_-(E'-H_0)U_-^*VU_-\delta(E'-H_0)dE' - \varepsilon\int\delta(E-H_0)U_-^*VU_-\gamma_-(H_0-E)dE$$

$$= \varepsilon\iint\gamma_-(E'-E)\delta(E-H_0)U_-^*VU_-\delta(E'-H_0)dEdE'$$

$$- \varepsilon\iint\delta(E-H_0)U_-^*VU_-\gamma_-(E'-E)\delta(E'-H_0)dEdE' = 0.$$

Hence the third term on the right of (II-1) merely cancels the second and third terms and we are left with our desired result

$$U_-^*\,U_- = I.$$

The relation $U_+^*\,U_+ = I$ is proved similarly.

Appendix III

Proof of $U = U_{\pm} M_{+}$

Let us prove $U = U_{-}M_{-}$. From equations (4.6) and (5.1) we have

$$\text{(III-1)} \quad U_{-}^{*}U = M_{-} + \varepsilon \int \gamma_{-}(E-H_{0})VU\ \delta(E-H_{0})dE + \varepsilon \int \delta(E-H_{0})U_{-}^{*}V\ \gamma_{-}^{*}(E-H_{0})dEM_{-}$$
$$+ \varepsilon^{2} \iint \delta(E-H_{0})U_{-}^{*}V\ \gamma_{-}^{*}(E-H_{0})\gamma_{-}(E-H_{0})VU\delta(E-H_{0})dEdE.$$

By a procedure similar to that used in Appendix II, one can show that the fourth term on the right of (III-1) cancels the second and third term. Hence

$$\text{(III-2)} \qquad U_{-}^{*}U = M_{-}.$$

Finally, on using the completeness relation (4.25) we have the desired result. The relation $U = U_{+}M_{+}$ is proved similarly.

Appendix IV

Alternative Uniqueness Theorem

Theorem: If the inverse of $\left[I + \varepsilon\Omega\, \eta(q-Q)\right]$ exists and is unique and if the inverse of Ω exists, then the only possible solution of (8.16) for K is given by (8.17).

Proof: From the triangularity properties of K we may write

$$K = \int_{q_o}^{q_1} \delta(q-Q)K\, \eta(q-Q)dq. \tag{IV-1}$$

Equation (8.17) may then be written

$$\int_{q_o}^{q_1} \delta(q-Q)K\, \eta(q-Q)\left[I + \varepsilon\Omega\, \eta(q-Q)\right]dq = -\int_{q_o}^{q_1} \delta(q-Q)\Omega\, \eta(q-Q)dq. \tag{IV-2}$$

Now K is to be a triangular operator such that $< q|K|q' > = 0$ for $q' > q$. Accordingly it must always be possible to represent K in the form (see (8.13))

$$K = \int_{q_o}^{q_1} \delta(q-Q)P\, \eta(q-Q)dq, \tag{IV-3}$$

where $< q|P|q' > = < q|K|q' >$ for $q' < q$. If Ω has an inverse we can always find an operator T such that

$$T = P\Omega\,. \tag{IV-4}$$

Hence the general expression for a triangular operator K can be written

$$K = \int_{q_o}^{q_1} \delta(q-Q)T\Omega\, \eta(q-Q)dq. \tag{IV-5}$$

It is our objective to obtain the most general operator T which will allow K to satisfy (IV-2). Having obtained this operator we shall obtain from (IV-5) the most general K which can satisfy this equation. Let us substitute (IV-5) into (IV-2) to obtain an equation for T. The left-hand side becomes

(IV-6) $$\int_{q_0}^{q_1} \delta(q-Q)\left[\int_{q_0}^{q_1} \delta(q'-Q)T\Omega\,\eta(q'-Q)dq'\right] \eta(q-Q)\cdot\left[I + \varepsilon\Omega\,\eta(q-Q)\right]dq$$

$$= \int_{q_0}^{q_1} \delta(q-Q)T\Omega\,\eta(q-Q)\,\eta(q-Q)\,\left[I + \varepsilon\Omega\,\eta(q-Q)\right]dq$$

$$= \int_{q_0}^{q_1} \delta(q-Q)T\Omega\,\eta(q-Q)\left[I + \varepsilon\Omega\,\eta(q-Q)\right]dq$$

$$= \int_{q_0}^{q_1} \delta(q-Q)T\,\left[I + \varepsilon\Omega\,\eta(q-Q)\right]\Omega\,\eta(q-Q)dq\ .$$

In (IV-6) we have used the relations

(IV-7) $$\delta(q-Q)\delta(q'-Q) = \delta(q-Q)\delta(q-q')$$

(IV-8) $$\eta(q-Q)\eta(q-Q) = \eta(q-Q).$$

Hence we have from (IV-2) an equation for T, namely

(IV-9) $$\int_{q_0}^{q_1} \delta(q-Q)T\,\left[I + \varepsilon\Omega\,\eta(q-Q)\right]\Omega\,\eta(q-Q)dq = -\int_{q_0}^{q_1} \delta(q-Q)\Omega\,\eta(q-Q)dq.$$

Equation (IV-7) is of the form

(IV-10) $$\int_{q_0}^{q_1} \delta(q-Q)A\eta(q-Q)dq = \int_{q_0}^{q_1} \delta(q-Q)B\,\eta(q-Q)dq,$$

where A and B are two operators. In terms of the Q-representation (IV-10) reads

(IV-11) $$< q|A|q' > \eta(q-q') = < q|B|q' > \eta(q-q'),$$

which means

(IV-12) $$< q|A|q' > = < q|B|q' >, \qquad \text{for } q' < q,$$

$$(\text{IV-13}) \qquad < q|A|q' > = < q|C|q' >, \qquad \text{for } q' > q,$$

where C is an arbitrary integrable operator. Let us take $C = B + D$ where D is arbitrary; then

$$(\text{IV-14}) \qquad < q|A|q' > = < q|B|q' > + < q|D|q' > \eta(q'-q),$$

or

$$(\text{IV-15}) \qquad A = B + \int_{q_0}^{q_1} \delta(q-Q) D \, \eta(Q-q) dq.$$

Equation (IV-15) is the most general solution of (IV-10) for A in terms of B. Let us identify A with $T\left[I + \Omega\eta(q-Q)\right]\Omega$ of equation (IV-9) and B with $-\Omega$. Then we have

$$(\text{IV-16}) \qquad T\left[I + \Omega\eta(q-Q)\right]\Omega = -\Omega + \int_{q_0}^{q_1} \delta(q-Q) D \; \eta(Q-q) dq,$$

from which

$$(\text{IV-17}) \qquad T = -\left[I + \epsilon\Omega\,\eta(q-Q)\right]^{-1} + \int_{q_0}^{q_1} \delta(q'-Q) D \, \eta(Q-q') dq' \, \Omega^{-1}$$

$$\cdot \left[I + \epsilon\Omega\eta(q-Q)\right]^{-1}.$$

This is the most general solution one can obtain for T. With this T one obtains the most general solution for K, on using (IV-5), namely

$$(\text{IV-18}) \quad K = \int_{q_0}^{q_1} \delta(q-Q) T \Omega\, \eta(q-Q) dq = -\int_{q_0}^{q_1} \delta(q-Q) \left[I + \epsilon\Omega\eta(q-Q)\right]^{-1} \Omega\, \eta(q-Q) dq$$

$$+ \int_{q_0}^{q_1} \delta(q-Q) \left[\int_{q_0}^{q_1} \delta(q'-Q) D \, \eta(Q-q') dq'\right] \Omega^{-1} \left[I + \epsilon\Omega\eta(q-Q)\right]^{-1} \cdot \Omega\eta(q-Q) dq.$$

Now the first term on the right is the solution which we obtained previously. The second term will be shown to be zero: it becomes

(IV-19)
$$\int_{q_0}^{q_1} \delta(q-Q) D\, \eta(Q-q) \Omega^{-1} \Omega \eta(q-Q) \left[I + \varepsilon \Omega \eta(q-Q) \right]^{-1} dq$$
$$= \int_{q_0}^{q_1} \delta(q-Q) D\, \eta(Q-q)\, \eta(q-Q) \left[I + \varepsilon \Omega \eta(q-Q) \right]^{-1} dq = 0$$

since

(IV-20)
$$\eta(Q-q)\, \eta(q-Q) = 0.$$

In (IV-19) we have used the fact that $\Omega \eta(q-Q)$ commutes with $\left[I + \varepsilon \Omega \eta(q-Q) \right]^{-1}$, a result obtained from equation (8.19).

We have thus shown that under our assumptions $\left[I + \varepsilon \Omega \eta(q-Q) \right]^{-1}$ and Ω^{-1} exist and are unique, and that hence the only solution for K is given by (8.17).

Appendix V

Alternative Proof that $U^{-1} = U_o$

Theorem: If the conditions on Ω are such that the equation $WU^* = U_o$ has the solutions U and U_o given by (8.23) and (8.24), and such that Ω^{-1} exists, then $U^{-1} = U_o$.

We shall first prove that

$$UU_o = I \tag{V-1}$$

or, what is the same, that

$$K + K_o = -\varepsilon\, KK_o. \tag{V-2}$$

It will be convenient to introduce the notation

$$A^{-1}(q) = I + \varepsilon\,\Omega\,\eta(q-Q), \tag{V-3}$$

$$A(q) = \left[I + \varepsilon\,\Omega\,\eta(q-Q)\right]^{-1}, \tag{V-4}$$

from which

$$A^{*-1}(q) = I + \varepsilon\,\eta(q-Q)\,\Omega \tag{V-5}$$

$$A^{*}(q) = \left[I + \varepsilon\;\eta(q-Q)\,\Omega\right]^{-1}. \tag{V-6}$$

Now in terms of $A(q)$ we have from (8.17) and (8.27)

$$K = -\int_{q_o}^{q_1} \delta(q-Q)A(q)\,\Omega\,\eta(q-Q)\,dq, \tag{V-7}$$

$$K_o = \int_{q_o}^{q_1} \eta(Q-q)A(q)\,\Omega\,\delta(q-Q)\,dq. \tag{V-8}$$

Hence

(V-9) $$KK_o = -\int_{q_o}^{q_1}\int_{q_o}^{q_1} dqdq' \; \delta(q-Q)A(q)\Omega\,\eta(q-Q)\cdot\eta(Q-q')A(q')\Omega\,\delta(q'-Q).$$

Now using the facts that

(V-10) $$\int_{q_o}^{q_1} \delta(q-Q)dq = I$$

and

(V-11) $$\begin{cases} \delta(q-Q)\,\eta(q'-Q) = \eta(q'-q)\delta(q-Q) \\ \delta(q-Q)\,\eta(Q-q') = \eta(q-q')\delta(q-Q), \end{cases}$$

we see that from (V-7) and (V-8) we can write

(V-12) $$K = -\int_{q_o}^{q_1}\int_{q_o}^{q_1} dqdq' \; \delta(q-Q)\,\eta(q-q')A(q)\Omega\,\delta(q'-Q),$$

(V-13) $$K_o = \int_{q_o}^{q_1}\int_{q_o}^{q_1} dqdq' \; \delta(q-Q)\,\eta(q-q')A(q')\Omega\,\delta(q'-Q).$$

Hence

(V-14) $$K+K_o = \int_{q_o}^{q_1}\int_{q_o}^{q_1} dqdq' \; \delta(q-Q)\,\eta(q-q')\Big[A(q')-A(q)\Big]\Omega\,\delta(q'-Q).$$

Now

(V-15) $$\begin{aligned} A(q') - A(q) &= A(q)\Big[A^{-1}(q)A(q') - I\Big] \\ &= A(q)\Big[A(q') - I + \varepsilon\Omega\,\eta(q-Q)A(q')\Big]. \end{aligned}$$

From (8.19)

(V-16) $$A(q') - I = -\varepsilon\Omega\,\eta(q'-Q)A(q') .$$

Hence

$$\text{(V-15a)} \qquad A(q')-A(q) = \varepsilon A(q)\Omega \left[\eta(q-Q)-\eta(q'-Q)\right] A(q').$$

Furthermore,

$$\text{(V-17)} \qquad \eta(q-q')\left[\eta(q-Q)-\eta(q'-Q)\right] = \eta(q-Q)\,\eta(Q-q').$$

Hence, after some obvious rearrangements we have

$$\text{(V-18)} \qquad \eta(q-q')\left[A(q')-A(q)\right] = \varepsilon A(q)\Omega\eta(q-Q)\,\eta(Q-q')A(q').$$

Substituting this expression in (V-14) we have, on comparing with (V-9),

$$\text{(V-19)} \qquad K+K_o = \varepsilon \int_{q_o}^{q_1}\int_{q_o}^{q_1} dq\,dq'\ \delta(q-Q)A(q)\Omega\eta(q-Q)\,\eta(Q-q')A(q')\Omega\delta(q'-Q) = -\varepsilon KK_o\,,$$

as required. We shall now prove

$$\text{(V-20)} \qquad U_oU = I.$$

Equation (V-20) is equivalent to

$$\text{(V-21)} \qquad WU^*U = I.$$

Since we are assuming that W has an inverse W^{-1}, equation (V-21) is equivalent to

$$\text{(V-22)} \qquad W^{-1} = U^*U.$$

From equation (8.23) we have

$$\text{(V-23)} \qquad U = \int_{q_o}^{q_1} \delta(q-Q)A(q)dq\,, \qquad U^* = \int_{q_o}^{q_1} A^*(q')\delta(q'-Q)dq'.$$

Thus, on using $\delta(q-Q)\delta(q'-Q) = \delta(q-Q)\delta(q-q')$ we obtain

$$\text{(V-24)} \qquad U^*U = \int_{q_o}^{q_1} A^*(q)\delta(q-Q)A(q)dq.$$

At this point it will be convenient to introduce derivatives of operators. We shall do so formally, although undoubtedly it is possible to justify the steps rigorously. On taking the derivative with respect to q of

(V-25) $$A(q)A^{-1}(q) = I$$

one obtains

(V-26) $$\left[\frac{d}{dq} A(q)\right]A^{-1}(q) + A(q) \left[\frac{d}{dq} A^{-1}(q)\right] = 0$$

or

(V-27) $$\frac{d}{dq} A(q) = -A(q) \left[\frac{d}{dq} A^{-1}(q)\right]A(q).$$

But from (V-3)

(V-28) $$\frac{d}{dq} A^{-1}(q) = \varepsilon\Omega \frac{d}{dq} \eta(q-Q)$$

$$= \varepsilon \Omega \delta(q-Q).$$

Thus on substituting (V-28) into (V-27) and integrating we get

(V-29) $$A(q) = C - \varepsilon \int_{q_o}^{q} A(q')\Omega \delta(q'-Q)A(q')dq',$$

where C is an operator to be determined by boundary conditions on A(q). We note that

(V-30) $$\eta(q_1 - Q) = I$$

$$\eta(q_o - Q) = 0,$$

which follows from the fact that

$$\eta(q_1-Q)|\emptyset > = \int_{q_o}^{q_1} |q > \eta(q_1-q)dq < q|\emptyset >$$

for any state $|\emptyset >$. Since $q_o \leq q \leq q_1$, $\eta(q_1-q) = 1$. Hence $\eta(q_1-Q)|\emptyset > = \int_{q_o}^{q_1} |q > dq < q|\emptyset > = |\emptyset >$, from which the first of equations (V-30) follows. The second is obtained similarly. Therefore,

(V-31)
$$A^{-1}(q_1) = W$$
$$A^{-1}(q_0) = I,$$

from which it follows that

(V-32)
$$A(q_1) = W^{-1}$$
$$A(q_0) = I.$$

Using the second boundary condition of (V-32) in (V-29), we have

(V-33)
$$A(q) = I - \varepsilon \int_{q_0}^{q} A(q')\Omega\, \delta(q'-Q)A(q')dq',$$

and on using the first of equations (V-32) we get

(V-34)
$$W^{-1} = I - \varepsilon \int_{q_0}^{q_1} A(q)\Omega\, \delta(q-Q)A(q)dq \;.$$

But from (8.26)

(V-35)
$$A(q)\Omega = \Omega\, A^{*}(q).$$

Hence on using (V-34) and (V-24) we have

(V-36)
$$W^{-1} = I - \varepsilon\Omega \int_{q_0}^{q_1} A^{*}(q)\delta(q-Q)A(q)dq = I - \varepsilon\Omega U^{*}U.$$

Now

(V-37)
$$WW^{-1} = (I + \varepsilon\Omega)W^{-1} = I$$

or

(V-38)
$$W^{-1} = I - \varepsilon\Omega W^{-1}.$$

Comparing (V-38) and (V-36), we see that

(V-39)
$$\Omega W^{-1} = \Omega U^{*}U,$$

or, if Ω has an inverse,

(V-40)
$$W^{-1} = U^{*}U$$

or finally

$$WU^{*}U = U_{o}U = I, \qquad \text{(V-41)}$$

which was to be shown.

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7 Eq. (4.11) add minus sign in exponent to read

$$\lim_{t\to+\infty} e^{iH_o t} e^{-iHt} u_-|\emptyset > = S|\emptyset >.$$

16 In Eq. (7.3) change integration variable to read-dq'···instead of-da'···

16 In Eq. (7.4) change last factor

to read $< q'|H,A;E,a >$

instead of $< H,A;E,a >$

23 In first line of Proof

read $u^{(1)} = I + \varepsilon K^{(1)}$

instead of $u^{(1)} = I + K^{(1)}$.

In second line of Proof

read $u_o^{(1)} = I + \varepsilon K_o^{(1)}$

instead of $u_o^{(1)} = I + K_o^{(1)}$

50 In title of Appendix III

read $u = u_\pm M_\pm$

instead of $u = u_\pm M_+$.